M A T H E M A T I C A L   E C O N O M I C S

No. 14(21) 2018

DECISION-MAKING UNDER RISK AND "STATISTICAL THINKING" IN THE 20TH CENTURY (SELECTED MODELS AND PERSONS)

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Abstract. The paper is the second part of the series of articles surveying chosen models of decision-making under "risky circumstances". The first segment concerned the earlier period of development of so-called "statistical thinking" (up to the times of J. Neyman and E. Pearson) and has been published elsewhere. These "twins" of papers as a whole, are intended as essays (consciously avoiding any formalization) to introduce the subsequent parts of the cycle – conducted in a more formal style. Several problems were discussed in the first part of the series. The leitmotifs, i.e. Bayesian vs. "orthodox" approaches, and the subjective vs. objective probability meaning are continued in this article, and developed towards the "modern needs and directions". The role of some outstanding scientists is stressed. The possibility of the unification of the different philosophies on the grounds of statistical decision theory (thanks to A. Wald and L.J. Savage) is noted. "Dynamic" or multistage statistical decision procedures will be also indicated (in contrast to "static, "one-shot" problems). The primary role in developing these ideas played by mathematicians A. Wald, L. Shapley, R. Bellman, D. Blackwell and H. Robbins (plus many others) is stressed.

The outline is conducted in a "historical perspective" beginning with F. Ramsey's work and finishing at H. Robbins achievements – as being very influential in the further development of the stochastic methodology. The list of models, to be discussed in the subsequent ("formal-mode") article/s, is added at the end of the paper. The central role in the notes is played by the "procession" of the prominent representatives of the field. The first "series" of them was presented in the previous part of the cycle. The subsequent (nine) are placed here. These scientists built the milestones of statistical science, "created its spirit," exquisitely embedding the subject in the "general stochastic world". The presentation is supplemented with their portraits. The author hopes that some keystones determining the line-up can be recognized in the course of reading. It is not possible to talk about mathematics without mathematics (formulas, calculations, formal reasoning). On the other hand − such beings as probability, uncertainty, risk can be, first of all, regarded as philosophic and logic in their heart of hearts (as well as being somewhat "mysterious"). So, it can turn out illuminating (sometimes) to reveal and to show merely the ideas and "their" heroes (even at the expense of losing the precision!). The role of the bibliography should also be stressed – it is purposely made so large, and significantly completes the presentation.

Keywords: statistics, risk, subjective probability, objective probability, frequentists, sequential analysis, stochastic approximation, stochastic game, empirical Bayes approach.

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JEL Classification: C180, C440

MS Classification: 62A01

DOI: 10.15611/me.2018.14.06

Probability does not exist
Bruno de Finetti [1937]

1. Introduction

The above – surely perverse – motto is placed fairly intentionally. Recalling this brilliantly provocative beginning of the famous de Finetti treatise on (subjective, "bet-style") probability (and, "on the occasion", statistics) [de Finetti 1937] is aimed to demonstrate the lack of unanimity about the meaning of the notion of "probability" and the very philosophical *status* of this category. Actually there are quite fundamental (eternal!) controversies in this field. The quoted – extreme one – formulation might be "softened" by adding just one restriction or condition: "in the sense of…" (simply by its relativization). The 20th century thoughts reveal the fairly wide range of various intermediate solutions to the problem (between the "extremes") providing the "space for compromises". At the same time, the consequent "obstinate", logically correct and mathematically complete (consistent) reasoning was also performed, which excluded getting closer alternative (concurring) points of view (from each position). The sides of the quarrel are far from goodwill in the subject and diminishing the differences. "Frequentists" postulate the "objective being" of the category (of probability) and argue mainly in the (fairly solidly grounded) spirit of laws of large numbers. "Subjectivists" supplementing it with "Bayesian thinking" in the area of statistics, show the logical inadequacy of "frequency-setting" for many cases. It should be stressed that the above questions were not "pub brawls", amateur disputes on the priority question of "which comes first, the chicken or the egg?" or "how many devils can be placed on a pinhead?" but serious polemics among prominent scientists, experts in the subject.

It might be useful to explain in the simplest possible way the main features of the subject in mind. The author wants to acknowledge the anonymous referee for the above suggestion. (Very) roughly speaking, the subjective (or individualistic) "philosophy of probability" assumes as a point of departure to further reasoning and constructions, the "nature" of this category as some "mixture of psychological and expert factor-components-determinants" leading to defining an (arbitrary, subjectivist) "measure of beliefs in occurring events in consideration", and (almost) equivalently "the degree of truth in judgments about a different state of the world" or "measure of indeterminacy (at least for the statistician, the decision maker – his/her degree of knowledge or ignorance in the matter). Such a subjective valuation of ("the importance" or a "a chance to occur") of the possible sets of states of Nature were formalized in the next stage of elaboration which, in turn, resulted in the forms: additive, countable additive or sub-additive (the famous Choquet's "capacities") set functions – however these (formal) consequences are not in "today's stream" of searching for (and identification of the) philosophical-logical sources and intuitive meaning of the discussed notion. Its, in a sense parallel statistical part postulates, roughly speaking, a kind of symmetry between observed values and unobserved elements of the "mechanics of randomness" – first of all, the parameters governing distributions of random elements (variables, vectors, processes). On the other hand, such an approach enables a maintenance of "informational symmetry" between the quantitative description of uncertainty concerning the so-called observables and "mysterious, hidden" parameters.

In the "objective-frequentialist world", all things proceed in another (opposite) way. The "objective measure of validity" (or the mass of") specific ("respective") events, subsets of a whole space of an elementary event (a possible, feasible state of Nature) are evaluated (are treated and estimated in just the opposite way): they exist as objective beings (characteristics of events) and very strong (first of all, from the viewpoint of logical "completeness and mathematical closedness") argument for justness and correctness of the above reasoning is provided by laws of large numbers-type (strict) mathematical proofs. The (evident) shortage of the construct makes the necessity of verifying the properness of the assignments of "weights" by repetitions which, in turn, excludes evaluations of non-repetitive (one shot, idionsyncratic) events. In the "statistical view" of this "orthodox" approach to probabilistic inference, the crucial idea concerns the undisputable constancy of unknown, estimated parameters: they in no way may be treated as "random" (in any sense) quantities.

Kołmogorov's perfect construction [Kolmogorov 1933] ("perfect" from mathematical point of view – "merely" and at the same time "so much"!) does not require any recommendation, just as the elaborated by generations Fisher-Neyman-Pearson(s) statistical systems (neglecting, at the moment, the differences among their approaches [Neyman, Pearson 1933]. But, on the other hand, ingenious (in the opinion of Milton Friedman [Friedman, Friedman

1998], among others) Leonard J. Savage [1954] wrote in 1961: "I proved the Bayesian argument in 1954. None of you find a flaw in the proof and yet you still deny it. Why?" [Savage 1961]. The most famous and fruitful turned out to be Herbert Robbins' idea of the so-called empirical Bayes approach to statistical problems, which may be seen as the concrete compromise position. At the same time, the methodology proposed by this author belongs to the area of "dynamic treatments" applied to decision-making under risk, where also belong such streams of (general) stochastic analysis as statistical sequential analysis, stochastic games and stochastic dynamic programming [Blackwell 1962], Markov decision processes, the whole theory of insurance risk processes [Bauvelinckx et al. 1990; Gerber 1979; Panjer, Willmot 1992], and stochastic modeling of the dynamics of financial processes [Föllmer, Schied 2004; Shreve 2004a; 2004b]] (on the small, as well as, on the large scale). Selected topics are referred further in the article. It is clear that the representatives chosen for the presentation are merely a sample of the "super population" of real creators of the discipline, so (with true regret!) such statisticians-probabilists as D. Blackwell [Blackwell 1947; Arrow et al. 1949] or S. Karlin [Karlin, Taylor 1998], and many others, have to be omitted.

The considerations are closed at Robbins' contribution, including also his established stochastic approximation concepts [Robbins, Monro 1951]. The shining development of modern stochastic finance and insurance models (which become more and more refined, engaging the theory of contemporary stochastic processes and, on the other hand, inspiring new theoretical constructions) will be discussed in the subsequent papers of the series. The same concerns the deeper analysis of the themes of large deviations [Embrechts et al. 1996] and rare events Chichilnisky [2010].

2. Frank Plumpton Ramsey and Richard von Mises – "two polar opposites in modern stochastic philosophy"

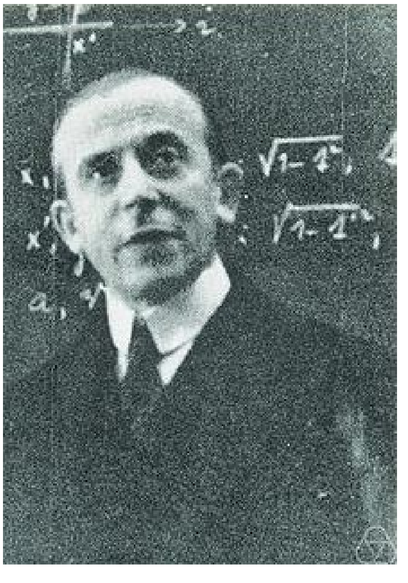
This part of the essay begins with a presentation of the somewhat "strange pair": of the philosopher cum logician cum economist cum mathematician, who tragically died at 27 and the philosopher cum mathematician cum physicist who lived to the age of 70. Both of them significantly contributed to the foundations of probability and statistics. Both are scientists of many talents and contributed importantly to the various branches of science, yet **their "visions" of the "proper" meaning of the crucial notion of "probability" are extremely different. They were Frank P. Ramsey (1903-1930) and Richard von Mises (1883-1953).**

Frank Plumpton Ramsey (1903-1930)

Source: [https://en.wikipedia.org/wiki/Frank_P._Ramsey].

Ramsey's interest was in the deep logic relations connected with fairly fundamental categories of science **("Truth and probability", 1926)** [Ramsey 1931]. The several fundamental questions concerned, named by Ramsey himself, respectively: The Frequency Theory, Mr Keynes' Theory, Degrees of Belief, The Logic of Consistency and the Logic of Truth. Very roughly speaking, he was "near" to **the subjective** (a la Bayesian) meaning of the notion of probability, and also admitted the role of utility (for quantifying risky phenomena). It should be also noted that "his probability" is, in a sense, integrated into the general logics system – one can "detect" in Ramsey's considerations an early version of fuzzy ideas in mathematics. Worthy of note is his famous polemic with John M. Keynes (on the roots of the matter probability) as well as differentiation of logical and statistical meanings of this notion. The notion of utility appeared in his pioneering work on the theory of economic growth, **"Mathematical theory of saving"** [Ramsey 1928].·This work turned out to be very **influential** in the emerging ideas of the game-theoretical approach to the economics contained in the monumental monograph by **John von Neumann and Oscar Morgenstern** [1944]. The importance of such **subjective (individualistic) "coping with risks"** has been revealed and demonstrated fairly recently. The contemporary history of the efforts of human intelligence in **dealing with risky situations of unparalleled, unusual or idiosyncratic types** (among others – some kind of catastrophic risk), when (classical, orthodox) statistical inference based mainly on multiple observations is impossible (simply for lack of such observations) proved evidently the above-mentioned role of **approaches to risks extending beyond the classic schemes** [Taleb 2007; Chichilnisky 2002].

Von Mises' approach to the foundations of probability are purely philosophical (despite his mathematical education and many valuable subjects in this area). His early, original work **"Probability, Statistics, and Truth"** [Mises 1928], developed the **"correct", objective**, or **"frequency" theory of probability.** The central idea of the theory was the so-called **"Kollektiv"**; it is worth noting that, years later, the main elements of this concept were **accepted even by A. Kolmogorov** [1965]. Similarly, in their seminal paper on the definition of probability (in a "Savagean wine"), **Anscombe and Aumann** [1963] combined **two different meaning of uncertainties**, which they termed " horse (logical) lotteries" and "roulette (physical) lotteries" , respectively: one-time (disposable) and repeatable. Thus **both** of the above "kinds of chance" turn out to be "real and reasonable" and **have to be taken into account when a behavior under risk is considered.**



Richard von Mises (1883-1953)

Source: [https://pl.wikipedia.org/wiki/Richard_von_Mises].

3. Three creators of modern statistical decision theory: towards the bridge linking the subjective approach and game-theoretical framework

Three scientists seem to be "responsible" for the **present form of statistical decision theory**, to be precise: its general shape. There are Abraham Wald, Bruno de Finetti and Leonard Jimmy Savage.

A. Abraham Wald (1902-1950**)** was a mathematician who worked in several fields of science of great importance to the whole cognitive processes of human kind in the first part of the 20th century. He especially contributed to such "hot" problems as the **general equilibrium** theory in economics (one of the Wald's papers provided the first correct, complete and elegant proof of the existence of an equilibrium in a "Walrasian" setting [Wald 1936], waiting for a formal solution from the Leon Walras epoch), linear programming methodology and econometrics (papers written in the 1930s). But the greatest achievement of this scientist turned out to be his concepts and results in "twofold areas" of statistical theory (which were in fact founded, built and developed by him). The first one was the **statistical sequential analysis** (general ideas and, the somewhat earlier, commonly known "Wald's quotient test"), the series seminal articles and the famous monograph **"Sequential Analysis" published in 1947** [Wald 1947]. The second contribution of great importance to the development of statistics as a whole, was the "sub-theory" of general statistics presented in another book of that author, which should be regarded as a pioneering monograph, featuring a new, fairly general look at the statistical methodology. The book appeared three years after "Sequential Analysis": It was entitled **"Statistical Decision Functions"** [Wald 1949]. The main ideas of statistical sequential analysis were: shortening the average time of observations and diminishing the cost of statistical research through reducing the "volume" of the sample. The number of observations ("atomic trials") is not stated before the experiment, on the contrary, it is a random variable ("random time"), depending on the successive "single" results, extending the information gathered in trials. The procedure goes step by step so its character may be classified as a dynamic one. The variety of "technical" (very important) details will be discussed in the previously announced article. At present, only one point should be signaled (on the more general aspect of multi-stage stochastic decision processes, which may be of the greatest importance): the ideas of **sequential analysis** are related to Bellman's dynamic programming [Bellman 1957] and **began the studies on the general theory of optimal (or optional) stopping**, developed further by Doob [1971] and many others. **The theoretical-decision approach to statistical problems enables the formal unifying of practice treatments of (at least) three main classes of problems**: hypotheses testing, (point) estimation and discriminant analysis, "embodying" them into the game-theoretic framework. On the other hand, such formalism joins and **reconciles (to a certain extent) Bayesian (or "subjective") and non-Bayesian philosophy in a statistical thinking.**

B. One of the leading ideologists, advocates and theorists of modern (close to the 20th cent. neo-Bayesians) **subjective probability theory** (as well as statistical inference in such a spirit) was **Bruno de Finetti (1906-1985).** This outstanding Italian scientist was a mathematician who specialized mainly in the probability theory, mathematical statistics and actuary field. Robert Nau [2001] writes: "The subjective theory of probability, which is now widely accepted as the modern view, is jointly attributed to de Finetti (1928/1937), Ramsey (1926/1931), and Savage (1954). Ramsey and de Finetti developed their theories independently and contemporaneously, and Savage later synthesized their work and also incorporated features of von Neumann and Morgenstern's (1944/1947) expected utility theory. All three authors proposed essentially the same **behavioristic definition of probability, namely that it is a rate at which an individual is willing *to bet* on the occurrence of an event".** This author would add the remark, that **"the bet" meaning of probability contains, *implicite*, the essence of notion of "arbitrage"** [Ellerman 1984] – some of the most important categories of the modern stochastic finance [Rybicki 2011].

At least two "scientific facts" seem to be worth mentioning when the very short life of de Finetti is sketched: the provocative statement **"probability does not exist"**, from which the author begins his treatise on the theory of probability [de Finetti1937]. This phrase means, according to the whole of de Finetti's reasoning (and the explanation) that probability does not exist in an objective sense. "It" exists only subjectively within the minds of individuals as "the *degree of belief* in the occurrence of an event attributed by a given person at a given instant and with a given set of information". The last sentence (in quotes) comes from the seminal paper (a kind of his **manifesto of subjective probability, published in 1937 in Annales de l'Institut Henri Poincaré): "La Prévision: ses lois logiques, ses sources subjectives"** [de Finetti 1937]. The second "thing" we are going to note is **the famous**

de Finetti's theorem about the connection between the so-called **interchangeability** (of infinite sequences of zero-one random variables) and **conditional independency** (of such sequences) [de Finetti 1931]. The problem appeared in a natural way when sampling is performed from "a source", which in itself is chosen randomly (according to the Bayesian framework) from some "super-population" of (hypothetical) sources. Its significance goes far beyond this context. Its subsequent generalizations in the spirit of probability theory (see, for instance [Rényi, Révész 1963]) or measure theory (in the case of infinite dimensional spaces, [Hewitt, Savage 1955]) reveal the extraordinary "intrinsic cognitive potential" of the construct originated by de Finetti.

C. Let us pass to the third person in this eminent trio presented in this paragraph. In "Tales of Statisticians"(2004) we can read that **Leonard Jimmie Savage (1917-1971)** was "probably **the most extreme advocate of a Bayesian, or in his word, 'personal', approach to probability questions that statistics has ever seen. Including Bayes himself**, who left his theorem unpublished due to his own doubts about it!". On the other hand Savage was (similarly to the scientists introduced earlier in the paragraph) a brilliant mathematician (differential geometry, measure theory). This circumstance caused that his interest (and works) in probability and statistics can be based on solid logic foundations, and he was able to engage refined mathematical tools which made his considerations more advanced and complete. All the papers and books by Savage can be described as very significant and influential contributions for the development of many branches of science. It should be stressed however that the majority of them arose to support the subjective Bayesian vision of an uncertain 'Savagean' world and justify the rules of statistical inferences binding in this world. Let us mention merely three commonly known papers of the highest importance (written in cooperation with other authors): **Application of the Radon-Nikodym Theorem to the Theory of Sufficient Statistics** (with P. Halmos [1949]), **"Utility Analysis of Choices Involving Risk"** (with M. Friedmann, [1948]), **"Symmetric measures on Cartesian products"** (with E. Hewitt [1955]).

Bruce Hill (2008, www.encyclopedia.com/doc/1G2-2830905312.html) writes: "**Savage's crowning achievement**, which grew out of the work of the greatest mathematicians and philosophers, including Blaise Pascal, James Bernoulli, Daniel Bernoulli, the Marquis de Laplace, Carl Friedrich Gauss, Henri Poincaré, Frank Ramsey, John von Neumann, and Bruno de Finetti,

was his book The Foundations of Statistics [1954]. Partly through the influence of von Neumann, who had developed the theory of games and formulated the basic ideas of decision theory, and partly through the influence of the English logician and mathematician Ramsey and the Italian mathematician and philosopher de Finetti, Savage developed in the first five chapters of his book the most complete version of the theory of subjective probability and utility that has yet been developed". Just the above quoted opinion (especially the last sentence of it) makes (or, rather, contains) the reason why we talk here about this scientist. The "Foundations…" **provided the philosophical grounds, the formal framework and the "operational" apparatus for making a "rational" decision under uncertainty (and risky situations), including numerous kinds of decisions under threat.**

The last (but not least!) remark on the "Savagean system" is the following one: departing from the some primitive formal postulates (reflecting, in turn, intuitive demands of rationality), he constructed – at the same time! – the function of subjective (additive merely – in contrast to the "proper" sigma additive) probability, together with the so-called **subjective expected utility.** The above **enabled quantifying concrete situations under uncertainty and measuring the consequences of occurrences of their hypothetical results. In a sense this work synthesized the ideas of von Neumann (games) and Ramsey-Wald-de Finetti (subjective probability and Bayesian statistics).** There existed however some **subtle differences in the meaning of ("personal") probability in the American style (Savage) and the European (Italian and French) schools (de Finetti, Ramsey)**. The photo below "documented" such a confrontation (on the highest scientific level!) during the Conference at Bressanone. But it is not the subject of this essay.

Passing to the end of this fragment we can mention the "complementary part" of Savage's studies on the area of behavior under risk. This is **his second seminal book, written together with L. Dubbins in 1965: "How to Gamble If You Must: Inequalities for Stochastic Processes"** [Dubins, Savage 1965]. One may call it "the second leg" of the theory of dealing with risky circumstances. It also makes **a step towards "dynamic stochastic procedures", joined with the learning and adaptive processes,** theory of martingales, repetitive plays "against Nature" and the theory of stopping times. All the above "sub-disciplines" deal with the "proper recognizing and active responding" to risks appearing in time.

Abraham Wald (1902-1950)

Source: [https://www.google.com/search?q=abraham+wald].

Bruno de Finetti (1906-1985) Leonard Jimmie Savage (1917-1971)

Source: [https://www.google.com/search?q=bruno+de+finetti;
https://www.google.com/search?q=leonard+jimmie+savage].

Bruno de Finetti and Leonard Jimmi Savage once again
This is the precious “paparazzi-style trophy”: two giants (“quasi-opponents”) “caught” at the same time! (Bressanone Scientific Meeting, 1961)

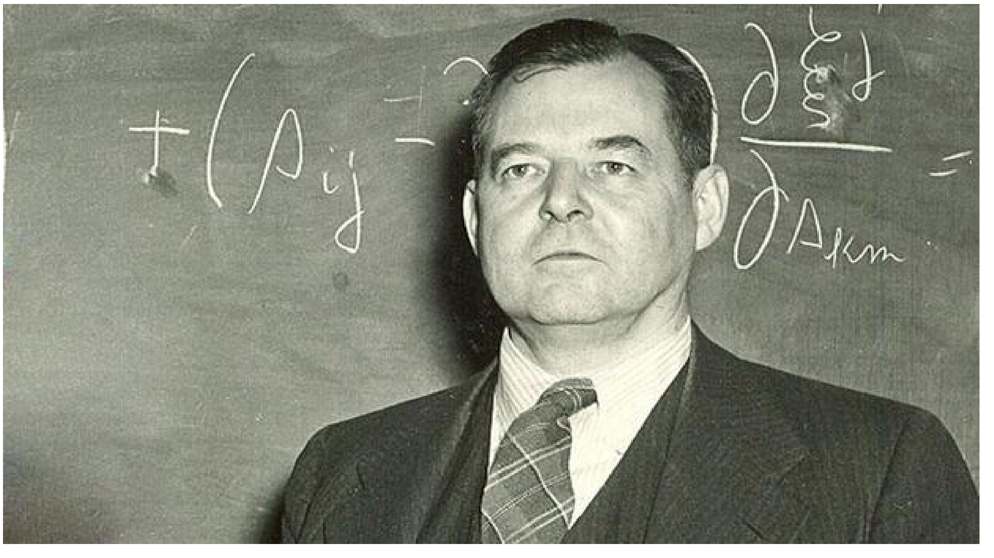
Source: [http://terzadecade.it/wp-content/uploads/2013/03/de-Finetti-e-Savage-a-Bressanone-1961.jpg].

The author feels obliged, once again, to remind that the article is intended to make merely a selection from the population of the prominent researchers contributing to development of stochastic formalism. However, several names (and works) should be also mentioned here as these authors significantly contributed to the subject, such as the following papers and books: Blackwell [1947], Blackwell and Fergusson [1968]; Blackwell and Girshick [1954]; Arrow, Blackwell, Girshick [1949]; Ferguson [1967]; DeGroot [1970], to mention but a few (and being fully aware that the list is far from being complete).

4. Harald Hotelling – the man who began the (scientific) concern about our future

The man who is regarded as the **pioneer** in the area of recognizing the environmental endangerment and began to search for the active strategy of managing non-renewable resources based on statistical concepts, was the American mathematician, statistician and economist **Harald Hotelling**

(1895-1973). In a sense he might be called "the earliest representative of the paradigm of sustainability in economics". H. Hotelling can also be regarded as the first "serious" continuator of the mathematical treatment of economic problems initiated by Leon Walras and the real precursor of the whole science discipline of **mathematical economics** (almost simultaneously with A. Wald).



Harald Hotelling (1895-1973)

Source: [https://www.google.com/search?q=harold+hoteling].

Again one may ask why is this scientist presented in this essay? The answer is not so difficult: **for his concern about our future safety and for preventing the danger of exhausting resources necessary for human life**. Besides his numerous significant ideas, stating problems and solving them in the area of statistics (early conceptions of confidence intervals, generalization of t-Students statistics to the multidimensional case, developing Carl Pearson's idea of principal components analysis), the most influential was the **seminal article from 1931 "The economics of exhaustible resources"** [Hotelling 1931]. S. Devarayan and A.C. Fisher [Devarayan, Fisher 1981] wrote: **"There are only a few fields in economics whose antecedents can be traced to a single, seminal article"** and concluded, that "The economics of exhaustible…" certainly can be included in the distinguished company of ground-breaking papers for the development of the whole modern branch of economics. His "application of the calculus of variations to the allocation of a fixed stock over time formed the basis of subsequent work on the subject "(Arrow). From the "practical economics" perspective (recommendations for management of resources), the idea of great importance is the so-called **"Hotelling rule"** claiming that the price of an exhaustible recourse must grow at a rate equal to the rate of interest, both along an efficient extraction path and in a competitive resource industry equilibrium. This rule determined many other later attempts at **diminishing the previously mentioned dangers of the encroaching environmental-type catastrophe of the total exhaustion of resources** and the death of the economic (as well as biological) life on the earth. (i.e. the **Hartwick rule** [Hartwick 1977]; **Dixit, Hoel-Hammond** [Dixit et al. 1980] rule and contemporary ideas of **hyperbolic discounting** when the evaluation of long-term projects is performed [Arrow 1999; Frederick, Loewenstein, O'Donoghue 2002; Gollier 2010].

5. The important steps towards the "dynamic-type" stochastic decision procedures (or several ingenious ideas, models and topics by Lloyd Shapley and Herbert Robbins)

There were several **golden-age periods of development of stochastic methods** during the last century, one of them in the 1950s – the "plus two-sided neighborhood" (including the late forties and early sixties). The series of most significant (in the common opinion of the probabilistic and statistical elites), **Berkeley Symposia** began then and three subsequent editions took place in that period [Symposium 1950; 1954; 1960]. At the same time, the variety of "scientific marriages" arose. Let us focus on two such **"associations": (a) stochastic models with dynamic programming methods, (b) statistics (especially in its "Bayesian slope or conceptualization") with the game theory.** Of course, all efforts made in the field can be seen as part of the (stochastic) optimization domain.

A. Professor Lloyd Shapley (1923-2016) was an American mathematician and economist, an eminent specialist in the **theory of games**. As is commonly known, the main objective of this theory is modeling the situations involving conflicts and/or cooperation among "sides of the subjects" and providing solutions maximizing common and/or individual levels of satisfactions, as well as ("dually") minimizing the endangerments accompanying these situations. Actually, Shapley was the co-creator of the most important notions and modern techniques of this theory, (see, for instance, the papers

[Shapley 1953a; 1953b; 1967; 1969]). "Since the work of von Neumann and Morgenstern in 1940s, Shapley has been regarded … as **the very personification of game theory**. With Alvin E. Roth, **Shapley was awarded the 2012 Nobel Memorial Prize in Economic Sciences** for the theory of stable allocations and the practice of market design"[1]. It is not a proper place here to explore the many-faceted achievements of Shapley made during his long life (he died in March 2016).

Therefore, we are only going to mention his **noteworthy, original concept from 1953, of the so-called stochastic game** [Shapley 1953a] (which one can describe, without a great exaggeration, as fairly sublime and which turned out to be extremely fruitful for the development of dynamic stochastic programming [Blackwell 1962] and its specification – Markov decision processes [Puterman 1994]). **In contrast to the "standard statistical games" (a term used synonymously with the "statistical decision problems") the essence of the stochastic game is in its repetitiveness,** according to the following scheme (in its simplest form) proposed by Shapley himself: "in the stochastic game the play proceeds step by step, from position to position, according to transition probabilities controlled jointly by the two players. It is assumed a finite number of positions, and finite numbers of choices at each position; nevertheless, the game may not be bounded in length. If, when at position *k*, the players choose their *i*-th and *j*-th alternatives, respectively, then with a probability greater than zero the game stops, while with certain probabilities the game moves to corresponding positions. It can be easily shown that the such a game ends with probability 1 after a finite number of steps… The payments accumulate throughout the course of play, depending on the actions chosen by players at given states" [Shapley 1953a].

Objectives of the players are maximizing their expected (total) payoffs (gains) resulting from the sequence of such one-step plays. The above description of the "mechanics and aims" of stochastic games have been performed in a very rough and informal way. According to the convention applied in this essay, basic mathematical properties of a stochastic game (including so called existence theorems) are omitted. Nevertheless, it reflects the **key idea of the discussed notion: first of all, its "dynamic, multistage" character, and the facts that the rules of moving (of systems) are governed by transition matrices and determined by the initial states,**

[1] See: [http://www.wsj.com/articles/lloyd-shapley-won-the-nobel-prize-for-economics-1923-2016-1458342678].

whereas the total gain resulting from the game depends also on a sequence of cubic-matrices of "single" payments in given circumstances of systems.

B. In the article published in the Review of the International Statistical Institute, J. Neyman [1962] stated that, in his opinion, **in the years 1950 and 1955 two breakthroughs appeared in the theory of statistical decision-making**. Neyman indicated the **"perpetrator"** (or, rather, originator), responsible for these scientific accidents, namely **Herbert Robbins (1915-2001**), regarded as one of the most prominent American mathematicians and statisticians of the 20th century. The mentioned breakthroughs were achieved thanks to two fruitful ideas formulated by Robbins and presented by him during two subsequent (The Second and Third) Berkeley Symposia on Mathematical Statistics and Probability, in 1950 and 1955, respectively: **"compound statistical decision problems" and "empirical Bayes approach to statistical decision-making"**, according to Robbins' original terminology (published in 1951 and 1956, respectively [Robbins 1951; 1956]).

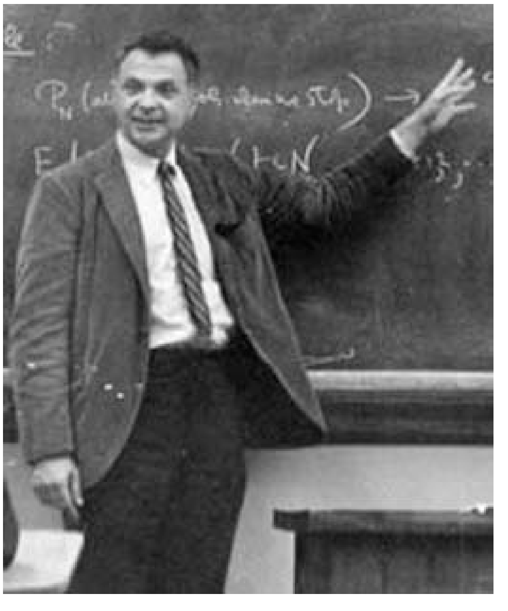
The compound decision theory concerns a sequence of independent statistical decision problems of the same form. Its basic thrust is the possibility of gaining a substantial reduction of total risk by allowing statistical procedures for the individual component problems to depend on the observations in the entire sequence. **It demonstrates, against naive intuition, that stochastically independent experiments are not necessarily "non-informative" to each other in statistical decision making** [Robbins 1951].

Five years later, Robbins developed the **empirical Bayes theory**. That construct concerns **experiments in which the unknown parameters are independent, identically distributed random variables with an unknown common prior distribution**. Empirical Bayes methodologies provide statistical procedures which approximate the ideal Bayes rule for the true model, so that the goal of the Bayesian inference is nearly achieved without specifying a priori! Such procedures usually perform well conditionally on the unknown parameters and provide – at the same time – solutions to compound decision problems [Robbins 1956].

It should be noted that the significance of "empirical Bayes philosophy" consists in (or is rooted in) undertaking "the mission impossible": bringing together two "polar opposite" methodologies – frequentist and Bayesian, "eliminating" the element of subjectivity from the latter (controversial arbitrariness, expressed in taking *a priori* distribution of an

estimated parameter as known). Contrary to this, empirical Bayes procedures perform in two-fold, simultaneous ways: subsequent coming (and gathered) information enriches the knowledge about the "really investigated" quantity and – at the same time – about the (unknown) distribution of the parameter ("mixed strategy" of Nature), on which depends the estimated value. It might be of some interest to note that "orthodox" Bayesians negate the above methodology from the fairly principal positions (philosophical as well as logical): they simply regard it as logically incorrect (Robbins himself was fully aware of the certain weakness of the "empirical Bayes" concept in this aspect) and treat its creator as a traitor while the idea as the "schism" (towards the "true Bayesism").

Lloyd Shapley (1923-2016)



Herbert Robbins (1915-2001)

Sources: [https://www.google.com/search?q=Lloyd+Shapley;
https://www.google.com/search?q=herbert+robbins+what+is+mathematics].

In addition, Herbert Robins became famous as the author **(together with Sutten Monro)** [Robbins, Monro 1951] **of the "stochastic approximation"** procedures, which generalized the classic Newton-Rawson ideas (to the stochastic environment) and were next extensively developed (in various directions), first of all by Kiefer and Wolfowitz [1952] and many others

[Nevel'son, Has'minskiĭ 1972; Kushner, Yin 1997]. The above mentioned authors outlined the idea in the following manner (in the quoted above, original paper): "Let $M(x)$ denote the expected value at level x of the response to a certain experiment. $M(x)$ is assumed to be a monotone function of x but is unknown to the experimenter, and it is desired to find the solution $x = \theta$ of the equation $M(x) = \alpha$, where α is a given constant. We give a method for making successive experiments at levels $x_1, x_2, \ldots$ in such a way that x_n will tend to θ in probability" [Robbins, Monro 1951]. **The stochastic approximation algorithms have <u>numerous</u> applications to the problems of minimizing the size of negative consequences or maximizing potential profits when acting under risk.**

6. Final remarks

In conclusion, the author would to present a summary of these considerations in two ways: to give a sketch of the discussed themes (justifying − by the way − some decisions made when composing the essay) and to announce that these issues are going to be formalized in the paper currently prepared for publication elsewhere.

A. First of all, let us mention the principal assumption: the departure point of the considerations on decision under risk (a bit more generally – in uncertain circumstances) is just the decision about the framework to be used for further analyses. The stochastic formalism has been proposed to this aim in the series of papers. Thus the piece of the history of "stochastic thinking and modeling", developed during the 20th century was presented in the article, but it was not aimed at, in any case, providing any systematic report or an overview (even within this narrow period!). We focused on following through a "competition of ideas" of various approaches "inside the stochastic paradigm". This concerns the very grounds of the meaning of the category "probability" itself as well as the principles of statistical inference as a tool for revealing stochastic mechanics governing various phenomena as precisely as possible and, at the same time, at the minimum "cost" (in its widest sense). At the same time, the theme of dealing with "environmental risks" was signaled. Following these considerations, the "dynamic face" of stochastic methodology revealed the sequential statistical analysis, the stochastic games, the empirical Bayes approach and the stochastic approximation models.

B. There seems to be the need to say "a word" about the choice of the "stopping time" for the process of the presentation of items and persons. The moment when the fragment ends is not actually random time: the logic of such a choice was derived, first of all, from the achieved level of advanced statistical methods. On the one hand, one may (in principle, not drastically erroneously!) feel that all important problems were then solved "with a reasonably satisfactory accuracy", so the time has just come to "close the chapter": the theory of statistical reasoning was actually completed and closed by "Foundations…" by L.J. Savage.

Yet at the same time, the appearance of signals of some breakthroughs in the sphere of "real" needs as well as the "theoretical" needs of stochastic modeling of "quality new phenomena" forced the search for "new" methods of description of processes, as well as the development of "new" methods on statistical inference on new classes of random processes. There were multi-stage procedures a la stochastic learning, adaptation and control, the large-scale financial-insurance stochastic models, further revisions and completing the axiomatic foundations of an uncertain world – strictly linked with its evaluation – through the (some kind of) utility functionals, indicating two crucial characteristics of the "random objects": their levels (magnitudes) together with the degree of riskiness. Therefore, the 1950s and 1960 may also seem to be the beginning of the modern era in stochastic modeling (such a long period, can be of course divided into sub-periods).

C. The approach used for the presentation of evolving and concurring ideas in the area of stochastic thinking and modeling was "through the portraits of their creators". In the author's opinion, this became acceptable thanks to the essayistic form of the paper (by the way, contrary to appearances, talking about formulas without formulas turned out not to be so easy). The idea of adding the "picture gallery" seems to be a good idea. Providing the formal concepts with "the human faces" brings them closer to the readers (especially those who are not professional statisticians) while showing the leaders of this discipline makes them more familiar. "Friendly" forms of transmission hope to enhance the desire of learning and further study of the subject.

D. The last question concerns the subsequent choice. We have to do with the "*embarass de richesse*" or *the excess of choice*. There does not exists precisely one "the only just, good choice". On the contrary, there were many prominent scientists (from the field of probability and statistics) in the history of the subject. One could (without any difficulty) construct (at least) two or three lists of The Great Absent (in the list placed in the outline). The basic criteria were: the evident, clear "cognitive interest" and "creative participation" in the discussion on principles of "stochastic visions of world", intuitions connected with category of "probability" and ways of formalization of the fundamental ideas and rules of reasoning. The above concern the "eternal" perspective, "originality of the current activity", specified regarding the concrete phase of development of the science as well as the force of impact on the further development of understanding and explaining the risk-bearing processes. It should be pointed out that the above qualification has to be made on the basis of commonly stated "historical" opinion, so the "subjective (author's) component" played merely a complementary role. Questions such as: "what about Harold Jeffreys [1939], Denis Lindley [1965], Harald Cramér [1946] or Hugo Steinhaus [1925; 1948; 1957]?" must not wait for the satisfactory answers. Their (undoubtedly significant) contributions to the science as a whole, have nothing to do with the fact of when some (author's own) concept was implemented.

E. Finally we outline the main points of the "list of items" which seem to be worth mentioning in a formal way in the next articles of the prepared series:

(i) general framework of the statistical decision problems;
(ii) the sequential statistical analysis models (classics);
(iii) the introduction to the (general) stopping times theory;
(iv) the Bayesian approach to the statistical decision problems;
(v) the hierarchical Bayesian models;
(vi) the empirical Bayes models;
(vii) the (finite sets of states and actions) stochastic game;
(viii) the simplest Markov decision process (discrete time);
(ix) the gambler's ruin problem and random walk models;
(x) the classic risk processes and some their generalizations.

In turn, the problems of the large deviations as well as extreme events and rare events, will be postponed to the forthcoming series of papers and referred to jointly with some questions concerning environmental and catastrophic risks, together with presenting the first "serious efforts" towards establishing the strict mathematical framework for the phenomena of the so-called Black Swans (introduced in the seminal book in 2002 [Taleb 2007] and in the article dated 2010 by Graziela Chichilnisky [Chichilnisky 2010]).

Bibliography

Anscombe F.J., Aumann R.A. (1963). *A definition of subjective probability*. The Annals of Mathematical Statistics. Vol. 34. No. 1, pp. 199-205.

Arrow K.J. (1999). *Discounting, morality and gaming*. [In:] P.R. Portney, J. Weyant (eds.). *Discounting and Intergenerational Equity*. Resources for the Future. Washington.

Arrow K.J., Blackwell D., Girshick, M.A. (1949). *Bayes and minimax solutions of sequential decision problems*. Econometrika 17. No. ¾, pp. 213-244.

Bauvelinckx T., Goovaerts M.J., Heerwarden van A.E, Kaas R. (1990). *Effective Actuarial Methods*. North-Holland. Amsterdam-New York-Oxford-Tokyo.

Bellman R. (1957). *Dynamic Programming*. Princeton University Press.

Blackwell D. (1947). *Conditional expectation and unbiased sequential estimation*. The Annals of Mathematical Statistics. Vol. 18. No. 1, pp. 105-110.

Blackwell D. (1962). *Discrete dynamic programming*. Ann. Math. Statist. No. 33, pp. 719-726.

Blackwell D., Ferguson T. (1968). *The big match*. Ann. Math. Statist. No. 39, pp. 159-163.

Blackwell D., Girshick M.A. (1954). *Theory of Games and Statistical Decisions*. John Wiley & Sons, Inc.

Chichilnisky G. (2002). *Catastrophic Risk*. [In:] A.H. El-Shaarawi, W.W. Piegorsch (ed.). *Encyclopedia of Environmetrics*. John Wiley & Sons, Ltd. Chichester UK. Vol. 1, pp. 274-279.

Chichilnisky G. (2010). *The foundations of statistics with black swans*. Mathematical Social Sciences. Vol. 59. Issue 2, pp. 184-192.

Cramér H. (1946). *Mathematical Methods of Statistics*. Princeton University Press.

de Finetti B. (1931). *Funzione caratteristica di un fenomeno aleatorio*. Atti della R. Academia Nazionale dei Lincei. Serie 6. Memorie. Classe di Scienze Fisiche. Mathematice e Naturale. No. 4, pp. 251-299.

de Finetti B. (1937). *La Prévision: Ses Lois Logiques, Ses Sources Subjectives*. Annales de l'Institut Henri Poincaré. Vol. 7, pp. 1-68. Translated (1964) in Studies in Subjective Probability. Kyburg Jr. and H.E. Smokler (ed.). John Wiley & Sons. New York.

DeGroot M.H. (1970). *Optimal Statistical Decisions*. McGraw-Hill.

Devarajan S., Fisher A.C. (1981). *Hotelling's "Economics of exhaustible resources": Fifty years later*. Journal of Economic Literature. Vol. 19. Issue 1, pp. 65-73.

Dixit A., Hammond A, Hoel M. (1980). *On Hartwick's rule for regular maximin paths of capital accumulation and resource depletion*. Review of Economic Studies.

Doob J.L. (1971). *What is a martingale?* Amer. Math. Monthly. Vol. 78. No. 5, pp. 451-463.

Dubins L.E., Savage L.J. (1965). *How to Gamble if You Must: Inequalities for Stochastic Processes*. McGraw-Hill Book Company. New York.

Ellerman D.P. (1984). *Arbitrage theory: A mathematical introduction*. SIAM Review. Vol. 26. No. 2, pp. 241-261.

Embrechts P., Kluppelberg K., Mikosh Y. (1996). *Modelling Extremal Events for Insurance and Finance*. Springer-Verlag. Berlin, Heidelberg, New York.

Feller W. (1970). *An Introduction to Probability Theory and Its Applications*. 3rd ed. Wiley.

Ferguson T. (1967). *Mathematical Statistics: A Decision Theoretic Approach*. San Diego.

Fourth Berkeley Symposium on Mathematical Statistics and Probability. June 20-July 30 1960. Statistical Laboratory of the University of California, Berkeley, https://projecteuclid.org/all/euclid.bsmsp, 1961, Proceedings of the Fourth Berkeley Symposium on

Mathematical Statistics and Probability, 1956, (ed.) J. Neyman, Berkeley, Calif., University of California Press, 1956.

Föllmer H., Schied A. (2004). *Stochastic finance*. *An introduction in discrete time*. de Gruyter Studies in Mathematics. Vol. 27. Walter de Gruyter & Co. Berlin.

Frederick S., Loewenstein G., O'Donoghue T. (2002). *Time discounting and time preference: a critical revive*. Journal of Economic Literature. Vol. 40, pp. 351-401.

Friedman M., Friedman R. (1998). *Two Lucky People: Memoirs.* The University of Chicago Press, Chicago, p. 146.

Friedman M., Savage L.J. (1948). *Utility analysis of choices involving risk*. Journal of Political Economy. No. 56 (4), pp. 279-304.

Gerber H. (1979). *An Introduction to Mathematical Risk Theory*. Huebner Foundation for Insurance Education. Philadelphia.

Gollier C. (2010). *Ecological discounting*. Journal of Economic Theory. No 145, pp. 812-829.

Halmos P.R., Savage L.J. (1949). *Application of the Radon-Nikodym theorem to the theory of sufficient statistics*. Ann. Math. Stat. No. 20, pp. 225-241.

Hartwick J. (1977). *Intergenerational equity and investing of rents from exhaustible economics.* American Economic Review. No. 53 (2), pp. 211-220.

Hewitt E., Savage L.J. (1955). *Symmetric measures on Cartesian products*. Trans. Amer. Math. Soc. No. 80, pp. 470-501.

Hill B. (2008). www.encyclopedia.com/doc/1G2-2830905312.html.

Hotelling H. (1931). *The economics of exhaustible resources*. The Journal of Political Economy. Vol. 39. Issue 2, pp. 137-175.

http://terzadecade.it/wp-content/uploads/2013/03/de-Finetti-e-Savage-a-Bressanone-1961.jpg.

https://en.wikipedia.org/wiki/Frank_P._Ramsey.

https://pl.wikipedia.org/wiki/Richard_von_Mises.

https://www.google.com/search?q=abraham+wald.

https://www.google.com/search?q=bruno+de+finetti.

https://www.google.com/search?q=harold+hotelling.

https://www.google.com/search?q=herbert+robbins+what+is+mathematics.

https://www.google.com/search?q=leonard+jimmie+savage.

https://www.google.com/search?q=Lloyd+Shapley.

https://www.wsj.com/articles/lloyd-shapley-won-the-nobel-prize-for-economics-1923-2016-1458342678.

Jeffreys H. (1939). *The Theory of Probability*. The Oxford University Press.

Karlin S., Taylor H.M. (1998). *An Introduction to Stochastic Modeling*. Third Edition. Academic Press.

Kiefer J., Wolfowitz J. (1952). *Stochastic estimation of the maximum of a regression function*. The Annals of Mathematical Statistics. No. 23 (3), pp. 462.

Kolmogorov A.N. (1933). *Grundbegriffe der Wahrscheinlichkeitsrechnung.* Springer-Verlag. Berlin.

Kolmogorov A.N. (1965). *Three approaches to the quantitative definition of information*. Problems Inform. Transmission. No 1(1), pp. 1-7.

Kushner H.J., Yin G.G. (1997). *Stochastic Approximation Algorithms and Applications.*

Lindley D. (1965). *Introduction to Probability and Statistics from a Bayesian Viewpoint*. 2 volumes. Cambridge University Press. Cambridge.

Mises von R. (1928). *Wahrscheinlichkeit, Statistik und Wahrheit*. 3rd German ed. Macmillan. New York. Translated (1957) as Probability, Statistics and Truth. 2nd revised English ed.

Nau R. (2001). *De Finetti was right: Probability does not exist*. Theory and Decision 51. Kluwer Academic Publishers, pp. 89-124.

Neumann von J., Morgenstern O. (1944). *Theory of Games and Economic Behavior*. Princeton University Press. Princeton.

Neyman J. (1962). *Two breakthroughs in the theory of statistical decision making*. Rev. Internat. Statist. Inst. No. 30, pp. 11-27.

Neyman J., Pearson E. (1933). *On the Problem of the Most Efficient Tests of Statistical Hypotheses*, Philosophical Transactions of the Royal Society A.

Nevel'son M.B., Has'minskiĭ R.S. (1972). *Stochastic Approximation and Recursive Estimation*. Translated by the Israel Program for Scientific Translations and B. Silver, Providence, RI: American Mathematical Society, 1973, 1976.

Panjer H., Willmot G. (1992). *Insurance risk models*. Society of Actuaries. Schaumburg IL.

Puterman M.L. (1994). *Markov Decision Processes*. Wiley Series in Probability and Mathematical Statistics. John Wiley and Sons. Chichester. England.

Ramsey F.P. (1928). *Mathematical theory of saving*. The Economic Journal. Vol. 38. No. 152, pp. 543-559.

Ramsey F.P. (1931). *Truth and Probability*. [In:] F.P. Ramsey, *The Foundations of Mathematics and Other Logical Essays*. Harcourt Brace. New York.

Rényi A., Révész, P. (1963). *A study of sequences of equivalent events as special stable sequences*. Publ. Math. Debrecen. No. 10, pp. 319-325.

Robbins H. (1951). *Asymptotically subminimax solutions of compound statistical decision problems*. Proceedings of the Second Berkeley Symposium on Mathematical Statistics and Probability. University of California Press. Berkeley and Los Angeles, pp. 131-148.

Robbins H. (1955). *An Empirical Bayes Approach to Statistical Decision Making*. Berkeley.

Robbins H. (1956). *An Empirical Bayes Approach to Statistics*. Proc. Third Berkeley Symp. on Mathematical Statistics and Probability. Vol. 1. University of California Press.

Robbins H., Monro S. (1951). *A stochastic approximation method*. The Annals of Mathematical Statistics. No. 22 (3).

Rybicki W. (1997). *Mathematics of utility and risk. Three papers on stochastic economy, finance and insurance*. Argumenta Oeconomica. No. 1(4), pp. 245-270.

Rybicki W. (2011). *The primer on arbitrage conceptions in economics: Their logics, roots and some formal models (historical and bibliographical notes)*. Mathematical Economics. No 7(4). The Publishing House of the Wroclaw University of Economics, pp. 173-198.

Rybicki W. (2014). *On Intuitive Meaning and Formal Defining the Notion of Risk*. [In:] S. Stanek, M. Popławski (ed.). *Decisions Under Endangerment*. Gen. Kościuszko Military Academy of Land Forces Publishing House, pp. 22-51.

Rybicki W. (2016). *On Decision Making Under Risk − The Outline of Selected Models*. [In:] D. Kuchta, M. Popławski, D. Skorupka, S. Stanek (Ed.). *Decisions in Situations of Endangerment: Research Development*. Wydawnictwo Wyższej Szkoły Oficerskiej Wojsk Lądowych im. generała Tadeusza Kościuszki. Wrocław, pp. 89-107.

Savage L.J. (1954). *The Foundations of Statistics*. John Wiley and Sons. New York. Revised edition 1972.

Savage L.J. (1961). *Bressanone Scientific Meeting.*

Shapley L.S. (1953a). *A Value for n-person Games*. [In:] H.W. Kuhn, A.W. Tucker (eds.). *Contributions to the Theory of Games.* Volume II.

Shapley L.S. (1953b). *Stochastic games*. Proc. Nat. Acad. Sci. USA. 39, pp. 1095-1100.

Shapley L.S. (1967). *On balanced sets and cores*. Naval Research Logistics Quarterly. Vol. 14, pp. 453-4.

Shapley L.S. (1969). *Utility comparison and the theory of games*. La Decision, pp. 251-263.

Shreve S. (2004a). *Stochastic Calculus for Finance I. The Binomial Asset Pricing Model*. Springer Finance Textbooks.

Shreve S. (2004b). *Stochastic Calculus for Finance II. Continuous Time Models*. Springer Finance Textbooks.

Spitzer F. (1976). *Principles of Random Walk*. Springer-Verlag.

Steinhaus H. (1925). *Definicje potrzebne do teorii gier i pościgu*. Myśl Akademicka. Lwów, pp. 13-14 (jednodniówka). Published in English in Naval Res. Logistic 1960, Quarterly. No. 7, pp. 105-107.

Steinhaus H. (1948). *Sur l'interpretation des r´esultats statistiques*. Colloquium 1925. No. 1. Mathematicum. Vol. 1, pp. 232-238.

Steinhaus H. (1957). *The Problem of Estimation*. The Annals of Mathematical Statistics. No. 28, pp. 633-648.

Symposium (1950). *Second Berkeley Symposium on Mathematical Statistics and Probability*. July 31-August12 1950. Statistical Laboratory of the University of California. Berkeley (Proceedings of the Second Berkeley Symposium on Mathematical Statistics and Probability. J. Neyman (ed.). University of California Press. Berkeley 1951).

Symposium (1954). *Third Berkeley Symposium on Mathematical Statistics and Probability*. December 1954 and July-August 1955. Statistical Laboratory of the University of California. Berkeley (Proceedings of the Second Berkeley Symposium on Mathematical Statistics and Probability. J. Neyman (ed.). University of California Press. Berkeley 1956).

Symposium (1960). *Fourth Berkeley Symposium on Mathematical Statistics and Probability*. June 20-July 30 1960. Statistical Laboratory of the University of California. Berkeley https://projecteuclid.org/all/euclid.bsmsp (Proceedings of the Fourth Berkeley Symposium on Mathematical Statistics and Probability, J. Neyman (ed.). University of California Press, Berkeley 1961).

Taleb N.N. (2007). *The Black Swan: The Impact of the Highly Improbable*. Penguin. London.

Wald A. (1936). *Über einige Gleichungssysteme der mathematischen Ökonomie*. Zeitschrift für Nationalökonomie. Vol. 7, pp. 637 [English trans. *On Some Systems of Equations of Mathematical Economics* (1951). Econometrica].

Wald A. (1947). *Sequential Analysis.* John Wiley and Sons. New York.

Wald A. (1949). *Statistical decision functions*. The Annals of Mathematical Statistics. Vol. 20. No. 2, pp. 165-205.