Optica Applicata, Vol. XV, No. 3, 1985

Application of weighted moments to image coding, decoding and processing. Part I. Reconstruction of an image from its weighted moments*

Adam Heimrath

Institute of Physics, Technical University of Wrocław, Wybrzeże Wyspiańskiego 27, 50-370 Wrocław, Poland.

The concept of a certain class of optical image digital representations presented in this paper is based on weighted optical moments. The current state of investigations on such nonorthogonal representations is reported. The optimal reconstruction procedure is given for the general case. The reconstruction-accuracy improvement by means of approximation based on Chebyshev polynomials is presented.

1. Introduction

An important problem in the optical/digital image processing and recognition is the choice of a proper mathematical representation of either intensity or complex amplitude distribution. There is no universal representation for all kinds of objects and operations. Such a representation, on the one hand, should be easily and accurately realized in the optical processor (for this reason it seems promising to apply the representations based on nonorthogonal transforms) and, on the other hand, supply the maximal amount of information in a limited quantity of digital data. The respective digital transformations and reconstruction procedures should be realized in a fast, simple and accurate way. For this reason, the orthogonal representations are more suitable, particularly when various operations on matrices are required [1].

The optical moments seem to assure the desired compatibility of both optical and digital processings [2–4]. These moments may be calculated in optical processors [5, 6]. Such representations may be orthogonalized in a particularly simple way. The relations among the moments and the image [7], the Fourier transform of the image [8], and the rotated, translated or rescaled image [9] are straightforward.

The aim of this paper is to present the generalized method of the optimal image reconstruction from the image weighted moments. Part II will present the possibilities of digital image processing connected with this representation.

---

* This work has been carried on under the Research Project M.R. I.5.

2. The reconstruction of the object from its moments

The moments of the distribution $f(x, y)$ are defined as

$$M^{pq} = \iint f(x, y) x^p y^p \, dx \, dy, \tag{1}$$

provided that the integral (1) is convergent.

In 1980 Teague proposed a method of reconstruction [9] based on the expansion into Legendre series. He assumed $f(x, y)$ in the following form:

$$f(x, y) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \lambda_{mn} P_m(x) P_n(y) \tag{2}$$

where $P_m(x)$ and $P_n(y)$ are the Legendre polynomials, which satisfy the orthogonalization conditions

$$\int_{-1}^{+1} P_m(x) P_{m'}(x) \, dx = \frac{2}{2m+1} \delta_{mm'}, \tag{3}$$

and $\delta_{mm'}$ is the Kronecker's delta. The coefficients λ_{mn} may be calculated as follows:

$$\lambda_{mn} = \frac{(2m+1)(2n+1)}{4} \int_{-1}^{+1} \int_{-1}^{+1} f(x, y) P_m(x) P_n(y) \, dx \, dy. \tag{4}$$

The relation between two series: $\{M^{mn}\}$ and $\{\lambda_{mn}\}$ is straightforward, since the Legendre polynomials may be expressed in the form of power series

$$P_m(x) = \sum_{i=0}^{m} c_{mi} x^i, \tag{5}$$

coefficients c_{mi} being given in [10]. Equations (4) and (5) yield

$$\lambda_{mn} = \frac{(2m+1)(2n+1)}{4} \sum_{i=0}^{m} \sum_{j=0}^{n} c_{mi} c_{nj} M^{ij}, \tag{6}$$

and the final approximation may be realized with truncated series (2). The fact that $f(x, y)$ is limited to the area $|x| \leqslant 1$ and $|y| \leqslant 1$ is not crucial, as $f(x, y)$ can always be rescaled to disappear outside that region.

3. The reconstruction accuracy

Teague presented in [9] some examples of reconstruction of simple pixel distributions from their moments. To examine the accuracy of this method we shall show the reconstruction of a step function (the 1-D case)

$$H(x) = \begin{cases} 0 & \text{for} \quad -1 \leqslant x < 0 \\ 1 & \text{for} \quad 0 \leqslant x < 1. \end{cases} \tag{7}$$

This elementary example illustrates the limitations of the method. In Table 1 we may see the rms error

$$\varepsilon = \left\{ \int_{-1}^{+1} [f(x) - \bar{f}(x)]^2 dx \Big/ \int_{-1}^{+1} [f(x)]^2 dx \right\}^{1/2} \tag{8}$$

(where $\bar{f}(x)$ denotes the approximation series (2) truncated and limited to 1-D), and the maximum deviation parameter

$$\psi = \max_{-1 \leqslant x \leqslant 1} |f(x) - \bar{f}(x)| \tag{9}$$

for various orders of the representation. Both parameters may be treated as the measures of reconstruction errors.

Table 1. Values of the rms error ε and maximum deviation ψ for the reconstructions of unit step function $H(x)$ from its usual moments with Legendre polynomials series of various orders P

P	3	5	7	9
ε	0.298	0.258	0.239	0.218
ψ	0.1875	0.1562	0.1367	0.1229

Table 2. Values of ε and ψ for the reconstructions of $H(x)$ from its moments with weighting function $w(x) = 1/\sqrt{(1-x^2)}$ corresponding to the Chebyshev approximation series of order P

P	3	5	7	9
ε	0.292	0.251	0.232	0.211
ψ	0.1001	0.0941	0.0909	0.0906

Figure 1 shows the reconstructions of $H(x)$ from representations of various orders p. Since the errors and deformations of $\bar{f}(x)$ with respect to $H(x)$ cannot be suppressed simply by increasing the order of representation and approximation series, it seems that the final effect may be improved by modifying either the reconstruction method or the representation.

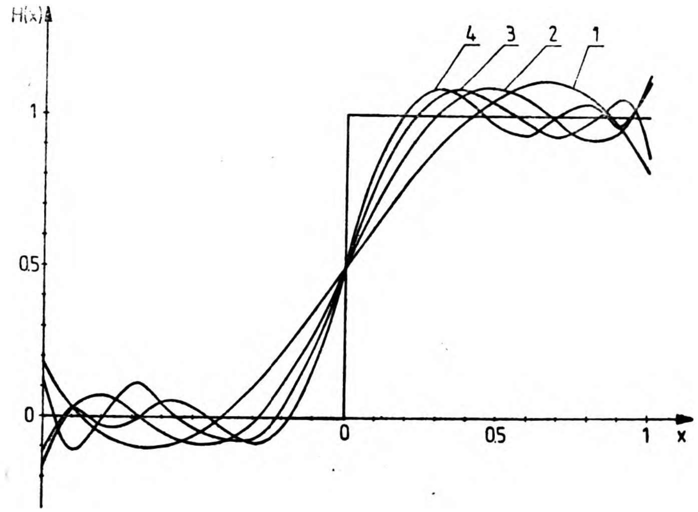

Fig. 1. Reconstructions of a unit step function $H(x)$ from its usual moments up to the order: $P = 3$ (curve 1), $P = 5$ (curve 2), $P = 7$ (curve 3), $P = 9$ (curve 4)

4. The representation of the function by its weighted moments

The proposed solution of the reconstruction problem is based on the application of the weighted distribution moments

$$M_w^{pq} = \iint f(x, y)w(x, y)x^p y^q dx\,dy \tag{10}$$

rather than usual moments (1). The $w(x, y)$ — weighting function — should preserve the convergence of the integral. The proposed solution is a generalization of Teague's approach.

Let us assume now that $f(x, y)$ may be approximated by $\bar{f}(x, y)$, where

$$f(x, y) \approx \bar{f}(x, y) = \sum_{i=0}^{P}\sum_{j=0}^{Q} a_{ij} W_i(x) W_j(y), \tag{11}$$

and the polynomials $\{W_i(x)\}$ and $\{W_j(y)\}$ are orthogonal over the approximation area Σ with the weighting functions $w_x(x)$ and $w_y(y)$, respectively

$$\int_{\Sigma} W_m(x) W_{m'} w_x(x)\, dx = a_m \delta_{mm'} \tag{12}$$

and, similarly, for $\{W_j(y)\}$.

The approximation is understood as a minimization process of

$$\sigma = \iint_{\Sigma} [f(x, y) - \bar{f}(x, y)]^2 w(x, y) dx dy$$
$$= \iint_{\Sigma} \Big[f(x, y) - \sum_{i=0}^{P} \sum_{j=0}^{Q} a_{ij} W_i(x) W_j(y)\Big]^2 w(x, y) dx dy \tag{13}$$

where $w(x, y) = w_x(x) w_y(y)$. The minimization means that

$$\frac{\partial \sigma}{\partial a_{ij}} = 0, \quad \begin{array}{l} \text{for } i = 0, \quad 1, \ldots, P, \\ \phantom{\text{for }} j = 0, \quad 1, \ldots, Q. \end{array} \tag{14}$$

Applying the conditions given in (14) to (13) we get a set of $(P+1)(Q+1)$ equations for a_{ij}

$$\iint_{\Sigma} f(x, y) w(x, y) W_p(x) W_q(x) dx dy$$
$$= \sum_{i=0}^{p} \sum_{j=0}^{q} a_{ij} \iint_{\Sigma} w(x, y) W_i(x) W_j(y) W_p(x) W_q(y) dx dy. \tag{15}$$

Noting the separability of $w(x, y)$ and orthogonality of both $W_i(x)$ and $W_j(y)$ we obtain $(P+1)(Q+1)$ independent equations

$$a_{pq} = \frac{\iint_{\Sigma} f(x, y) w(x, y) W_p(x) W_q(y) dx dy}{a_p a_q}. \tag{16}$$

The important feature of the reconstruction based on orthogonal polynomials is that the increasing reconstruction order is not accompanied with any change of the coefficients $\{a_{ij}\}$ calculated previously for lower orders. This accounts for the independence of all a_{pq} in (16). In order to calculate the integrals from (16) let us assume the expansion of each polynomial $W_p(x)$ and $W_q(y)$ into the power series

$$W_p(x) = \sum_{k=0}^{p} c_{pk} x^k, \qquad W_q(y) = \sum_{l=0}^{q} c_{ql} y^l. \tag{17}$$

In fact, most of the polynomials commonly used for approximation purposes may be expressed in this way. Application of these series to the integrals in (16) yields

$$a_{pq} = \frac{1}{a_p a_q} \sum_{k=0}^{p} \sum_{l=0}^{q} c_{pk} c_{ql} \iint_{\Sigma} f(x, y) w(x, y) x^k y^l dx dy. \tag{18}$$

The term-by-term integration of the series is permissible, due to the limited area of integration and convergence of the integrands. Finally, if we assume $f(x, y) = 0$ beyond Σ, then the integration on the right-hand side may be extended over the whole (x, y) plane. The integrals become identical with the respective weighted moments (10). This assumption is justified by the conditions of moment's existence (i.e., the integral convergence).

The final result is a simple expression of approximation coefficients in terms of weighted moments

$$a_{pq} = \frac{1}{a_p a_q} \sum_{k=0}^{p} \sum_{l=0}^{q} c_{pk} c_{ql} M_w^{kl}. \tag{19}$$

Setting $W_p(x) = P_p(x)$, $W_q(y) = P_q(y)$ and $w(x, y) = 1$ we get the solution proposed by Teague.

5. The application of orthogonal Chebyshev polynomials

Definition (10) generates a class of nonorthogonal representations based on weighted moments. Equations (11) and (19) make possible the reconstruction of the original distribution, provided that a set of proper orthogonal polynomials is known for given $w(x, y)$ and Σ.

In order to verify the idea of weighted moments' representation, the distribution $H(x)$ (Eq. (7)) was reconstructed from various representations with different $w(x, y)$.

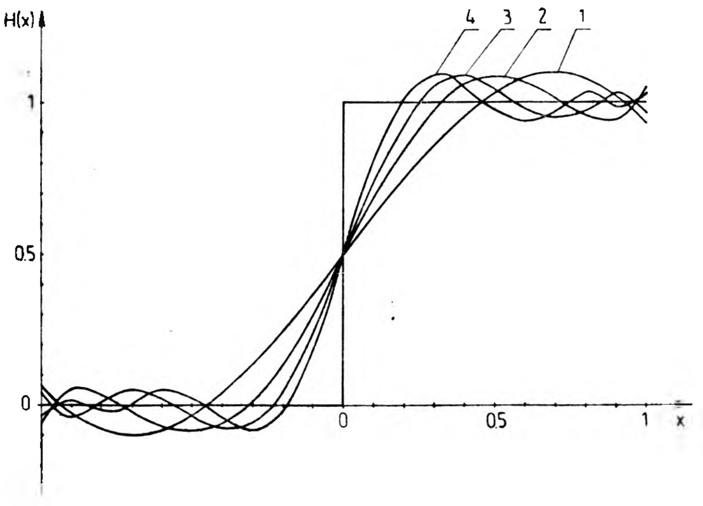

Fig. 2. Reconstructions of $H(x)$ from its moments with a weighting function $w(x) = 1/\sqrt{(1-x^2)}$; P — the maximum order of the moments involved ($P = 3$ (curve 1), $P = 5$ (curve 2), $P = 7$ (curve 3), $P = 9$ (curve 4))

Table 2 gives the results (parameters ε and ψ) for the reconstruction of $H(x)$ from the representation with weighting function $w(x) = 1/(1-x^2)^{1/2}$, corresponding to the orthogonal set of Chebyshev polynomials $T_i(x)$, where [10]

$$\int_{-1}^{+1} T_m(x)T_{m'}(x)\,\frac{1}{(1-x^2)^{1/2}}\,dx = \begin{cases} 0 & \text{for } m \neq m', \\ \pi & \text{for } m = m' = 0, \\ \pi/2 & \text{for } m = m' \neq 0, \end{cases} \tag{20}$$

and similarly for $\{T_j(y)\}$. The reconstructed distribution is shown in Fig. 2. The improvement is due to the well-known properties of Chebyshev polynomials, widely applied in the solutions of approximation problems (see, for example, [11]).

6. The examples of two-dimensional reconstructions

Further investigations included the reconstructions of non-complicated 2-D distributions. Figure 3 shows a lateral view of the distribution $f(x, y) = \text{rect}(x)\,\text{rect}(y)$ reconstructed from its moments up to the order 10 +10, with weight-

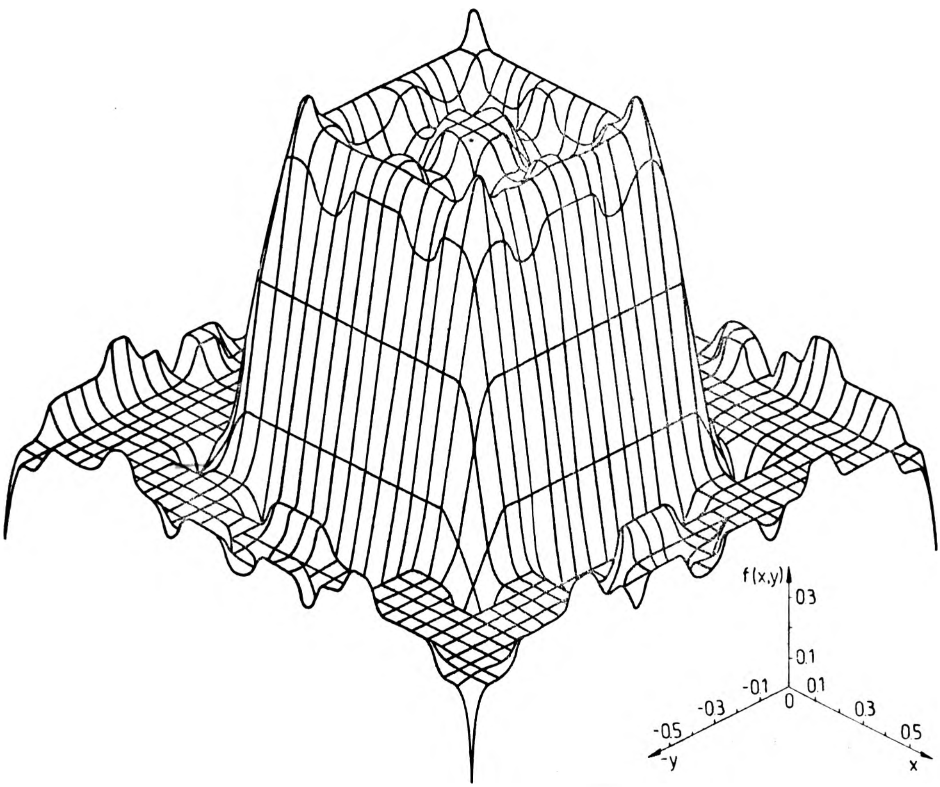

Fig. 3. Side view of the 2-D reconstruction of $f(x, y) = \text{rect}(x)\,\text{rect}(y)$ from its usual moments up to the order 10+10

ing function $w(x, y) = 1$. Figure 4a shows the cross-section of this reconstruction along the axis $x = y$ (referred to as r), compared with the reconstructions of lower orders. The respective ε and ψ values (the 2-D analogies of Eq. (8) and Eq. (9)) are given in Table 3a. The similar cross-sections of the same object reconstructed from its moments with weighting function $w(x, y) = 1/[(1-x^2) \times (1-y^2)]^{1/2}$ are shown in Fig. 4b. Parameters ε and ψ for these reconstructions are given in Table 3b. The case of a smooth function $f(x, y) = \exp(-x^2) \exp(-y^2)$ is shown in Figs. 5a, b and in Table 4a, b.

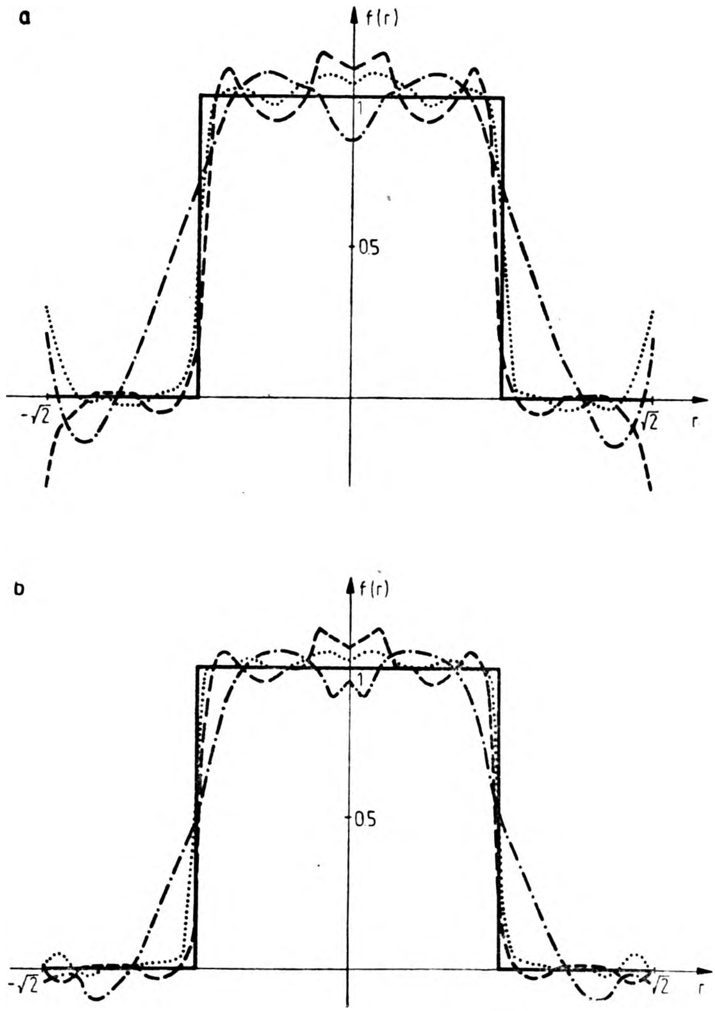

Fig. 4. Cross-sections ($x = y$) of $f(x, y) = \text{rect}(x)\,\text{rect}(y)$ from its moments: (a) usual, (b) weighted with $w(x, y) = 1/\sqrt{(1-x^2)(1-y^2)}$, with maximum order $P+P$ ($r = \pm\sqrt{x^2+y^2}$; ——— accurate values, ... $P+P = 9+9$, – – – – – – $P+P = 7+7$, — · — · — · — $P+P = 5+5$)

Table 3a. Values of ψ and ε for the 2-D reconstructions of $f(x, y) = \text{rect}(x)\text{rect}(y)$ from its usual moments up to the order $P+P$

P	3	5	7	9
ψ	0.531	0.312	0.183	0.122
ε	0.673	0.425	0.301	0.228

Table 3b. Values of ψ and ε for the 2-D reconstructions of $f(x, y) = \text{rect}(x)\text{rect}(y)$ from its moments with weighting function $w(x, y) = 1/\sqrt{(1-x^2)(1-y^2)}$ up to the order $P+P$

P	3	5	7	9
ψ	0.472	0.271	0.104	0.073
ε	0.551	0.402	0.152	0.130

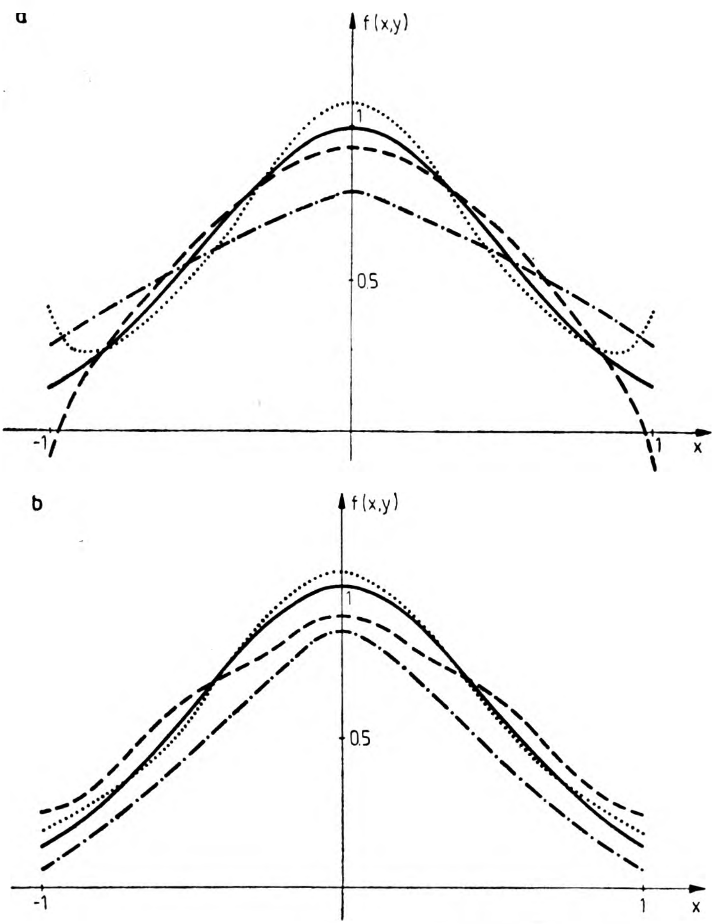

Fig. 5. Cross-section (along x axis) of $f(x, y) = \exp(-x^2)\exp(-y^2)$ from its moments: (a) usual, (b) weighted with $w(x, y) = 1/\sqrt{(1-x^2)(1-y)^2}$, with maximum order $P+P$ ($y = 0$); accurate values, (—— ...... $P+P = 7+7$, – – – – $P+P = 5+5$, – · – · – $P+P = 3+3$)

Table 4a. Values of ψ and ε for the 2-D reconstructions of $f(x, y) = \exp(-x^2)\exp(-y^2)$ from its usual moments up to the order $P+P$

P	3	5	7	9
ψ	0.393	0.272	0.192	0.081
ε	0.532	0.373	0.167	0.158

Table 4b. Values of ψ and ε for the 2-D reconstructions of $f(x, y) = \exp(-x^2)\exp(-y^2)$ from its moments with a weighting function $w(x, y) = 1/\sqrt{(1-x^2)(1-y^2)}$ up to the order $P+P$

P	3	5	7	9
ψ	0.364	0.209	0.161	0.072
ε	0.301	0.123	0.107	0.051

7. Conclusions

As it was shown above, the accuracy of the reconstruction of an object from its nonorthogonal, moment-based representation may be improved by the introduction of a weighting function, which enables the choice of proper approximating polynomials. In the case of the weighting function $w(x, y) = [(1-x^2)(1-y^2)]^{-1/2}$ which implies the approximation with Chebyshev polynomials, the improvement is obvious, mainly at the edges of the reconstructed area. Since the information contained at such a representation is — for the same order of representation — more precise when compared with that for usual moments, this representation seems to be a better discrimination space for image classification. The calculation of optical weighted moments requires the adoption of the processor presented in [5].

References

[1] Anrews H. C., *Two-Dimensional Transforms*, [in] *Picture Processing and Digital Filtering*, T. S. Huang (Ed.), Springer-Verlag, Berlin 1979.

[2] Caufield H. J., Haimes R., Casasent D., Opt. Eng. **19** (1980), 152.

[3] Hu M. K., IRE Trans. Inform Theory IT–**8** (1962), 179.

[4] Casasent D., Psaltis D., Proc. of SPIE **201** (1979), 107.

[5] Casasent D., Cheatham L., Fetterly D., Appl. Opt. **21** (1982), 3292.

[6] Teague M. R., Appl. Opt. **19** (1980), 1353.

[7] Duda R., Hart P., *Pattern Classification and Scene Analysis,* John Wiley and Sons, New York 1973.

[8] Bracewell R., *The Fourier Transform and its Application,* McGraw-Hill Co., New York 1965.

[9] Teague M. R., J. Opt. Soc. Am. **70** (1980), 920.

[10] Courant R., Hilbert D., *Methoden der Mathematischen Physik,* Springer-Verlag, Berlin 1968.
[11] Ralston A., *A First Course in Numerical Analysis,* McGraw-Hill Co., New York 1965.

Received October 8, 1984

Применение взвешенных моментов для кодирования, декодирования и преобразования изображений. Часть I. Реконструкция изображения но его взвешенным моментам

Дискутируется некоторый класс численных представлений оптического изображения, основанных на взвешенных моментах распределения напряжения или комплексной амплитуды. Обсуждено существующее до сих пор состояние исследований этого типа неортогональных представлений. Дана оптимальная процедура-реконструкции для общего случая. Доказано повышение точности реконструкции при применении процедуры, основанной на аппроксимации многочленами Чебышева по сравнению с используемыми до сих пор многочленами Лежандра.