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ANALYTICAL AND NUMERICAL SOLUTION OF EQUATIONS OF THE MATHEMATICAL MODEL OF TRADITIONAL RESERVOIR

The results of the tests of traditional reservoirs have been completed with a method for determining the maximal level of sewage storage; this method has been based on the assumption that there is a time in the storage phase after which the nonlinear functions describing fluctuations of sewage flow in the reservoir reach their maximum. Identifying the computational dependence of the coordinates of this characteristic point on the curve makes it possible to determine the essential design parameters for a given sewage inflow.

DENOTATIONS

AP – horizontal cross-sectional area of a flow chamber in a multichamber reservoir, m^2;

Fzr – reduced urban drainage basin area, ha;

h – sewage fill height in a traditional reservoir calculated from the outflow channel axis, m;

$h_0, h_1, h_2, ..., h_k$ – assumed values of possible maximum fill for a designed reservoir in the range $3.6 \geq h_i \geq 0.8$ m, but most commonly $3.0 \geq h_i \geq 1.2$ m;

hm – standard fill height of a traditional reservoir, m;

hp – sewage fill height in overflow chamber measured from outflow channel axis to overflow edge position in dual chamber reservoir, m;

H – sewage height in reservoir storage chamber measured from outflow channel axis, m;

QA – sewage inflow to reservoir, dm^3/s;

QA(TM) – maximum sewage inflow to reservoir from design storm for sizing a conventional ZK reservoir at $Td = TM$, dm^3/s;

QA(t) – instantaneous sewage flow in channel at time *t*, dm^3/s;

QO – sewage runoff from reservoir, variable over time, dm^3/s;

t – time, s;

Te – time after which reservoir is completely filled, corresponding to maximal sewage storage in reservoir, s;

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Tp – rainfall duration equal to inflow from the furthest basin point used to size the system by the method of maximal intensity, min;
TM – design storm duration for sizing a traditional single-chamber reservoir, min;
V – required capacity of reservoir relieving hydraulic conditions determined by a given method, m^3;
V_i, V_j – capacities of a given reservoir chamber for analytically determined fill heights H_i, H_j, m^3;
VK – capacity of single-chamber traditional reservoir, m^3;
c – frequency of storm design for the purpose of sizing the sewage system, years;
C, *D*, *K*, *u*, u_0, u_1, u_2 – constants of differential equations for describing sewage storage in traditional single-chamber reservoirs;
B, *B*1, *B*2, *C*1, *C*2, *C*6, *C*8 – constants of differential equations for sewage storage in traditional reservoirs;
g – acceleration of gravity, $m \cdot s^{-2}$;
H – mean annual rainfall, mm;
n – exponent for formulae used to calculate design storm runoff to sewage system;
α_c, α_H, α_n – correlation coefficients;
β – sewage flow reduction factor in reservoir, $\beta = QO \max \cdot QA(Tp)^{-1}$;
μ – factor for sewage runoff to outflow channel.

1. INTRODUCTION

Theoretical foundations that have been developed to describe sewage accumulation in storage reservoirs reflect different levels of complexity of mathematical procedures. The analytical and numerical solutions of the equations that describe the filling and evacuation of sewage in traditional, single-chamber reservoirs have been presented in a number of publications [1]–[4].

Sewage flow balance in the overflow chamber during the filling phase is established in a way similar to that of sewage accumulation in traditional reservoirs, but it is created as long as the sewage level does not exceed the height of the overflow barrier. The overflow chamber controls sewage flow inside the chambers of multichamber reservoirs. The level of the sewage in this chamber determines the conditions of the outflow from the reservoir as well as the participation of various components in the system during the filling and evacuation of individual chambers in multichamber storage reservoirs.

2. ANALYTICAL SOLUTION OF FLOW BALANCE EQUATIONS

The analytical solution of differential flow balance equations for traditional reservoirs makes it possible to obtain a particular non-homogeneous equation representing the first range of variation in the sewage inflow when $QA2 > QA1$ and to solve general equations (2) and (3) in the form of implicit functions [5]. Referring to the results of the test conducted so far, a general solution has been obtained for equation (1) that describes sewage flow balance during the first phase of filling the overflow chamber in a multichamber reservoir [5]:

$$\frac{dh}{dt} = C1 \cdot t - C2 \cdot h^{0.5} \quad \text{for} \quad QA2 > QA1 \ \text{(figure 3b)}, \tag{1}$$

$$\frac{dh}{dt} = C6 - C2 \cdot h^{0.5} \quad \text{for} \quad QA2 = QA1 \ \text{(figure 3c)}, \tag{2}$$

$$\frac{dh}{dt} = C8 - C1 \cdot t - C2 \cdot h^{0.5} \quad \text{for} \quad QA2 < QA1 \ \text{(figure 3d)}. \tag{3}$$

Equation (1) should be included among ordinary non-homogeneous first-order equations and it has been reduced to an equation of distinct variables [6], [7]. This equation allows a precise description of the filling levels in the overflow chamber of multichamber reservoirs for the initial condition $h = 0$ at $t = 0$ and $t(0, Tpp\rangle$ in the filling range of $h \leq hp$, where Tpp denotes the time it takes for the filling level in the chamber to reach $h = hp$. Equation (1) is solved by reducing it to simple fractions and by making some necessary transformations. The final form of the equation being solved is:

$$\left|h^{0.5} - u\right|^{u} = (h^{0.5} - u_0)^{u_0} \cdot B, \tag{4}$$

where:

$u = 0.5[(C^2 + 4 \cdot C1)^{0.5} - C2]$, $u_0 = -0.5[(C^2 + 4 \cdot C1)^{0.5} + C2]$,
$C = QA\,(2\,AP\,Tp)^{-1}$, $D = (2\,g)^{0.5} f\,\mu\,(2\,AP)^{-0.5}$.

Equation (4) has the form of an implicit function and is valid for the first range of inflow variation when $QA2 > QA1$ for the filling levels $hp \geq h \geq 0$, i.e., up to the edge of the highest level of the lowest overflow barrier in a multichamber reservoir. It is possible to establish an integration constant B from the condition that should be fulfilled in order to obtain an extreme function value, while a necessary condition for the existence of the extreme value of the implicit function is as follows:

$$F'(t) = 0 \quad \text{and} \quad F(h^{0.5}, t) = 0.$$

Because of their complexity, the other differential flow balance equations that describe sewage accumulation in multichamber reservoirs have been solved numerically. The methods used for solving equations and sets of differential equations can be found in the mathematical library BLCKDQ [7]. Appropriate methods turned out to be extrapolation–interpolation procedures with error check [7], [8] combined at the beginning with the Picard method [6].

3. METHOD FOR ESTABLISHING MAXIMUM SEWAGE FILL IN TRADITIONAL RESERVOIR

In order to determine the coordinates of the point M, at which the function describing the storage capacity of the reservoir reaches the maximum filling level, we have to establish the extreme limits of the function, taking into account the complexity of the inflow conditions. Sewage accumulation in the range of inflow variation when $QA2 < QA1$ is described by two different equations [1], [2], depending on an algebraic sign of the expression $D^2 - 4C$.

The coordinates of the point $M1$ correspond to the extreme value of function (5) when $D^2 - 4C > 0$

$$\frac{u_1 - D}{C(u_2 - u_1)} \ln\left|\frac{K - C \cdot t}{h^{0.5}} - u_1\right| - \frac{u_2 - D}{C(u_2 - u_1)} \ln\left|\frac{K - C \cdot t}{h^{0.5}} - u_2\right| - \frac{1}{2 \cdot C} \ln|h| - \ln B1 = 0 \quad (5)$$

and they can be determined from the condition that should be fulfilled in order to obtain the maximum extreme value of this function in the form of equations (6) and (7):

$$|hm1| = |D - u_1|^{\frac{2(u_1 - D)}{u_2 - u_1}} |D - u_2|^{\frac{-2(u_2 - D)}{u_2 - u_1}} B1^{-2C}, \quad (6)$$

$$Te1 = \left\{ K - D\left[|D - u_1|^{\frac{u_1 - D}{u_2 - u_1}} \cdot |D - u_2|^{\frac{D - u_2}{u_2 - u_1}} \cdot B1^{-C} \right] \right\} C^{-1}. \quad (7)$$

Similarly, the coordinates of the point $M2$ can be determined from the condition that should be fulfilled in order to obtain the maximum extreme value of function (8)

$$\frac{D}{C(4C - D^2)^{0.5}} \operatorname{arctg}\left[\frac{\frac{K - Ct}{h^{0.5}} - \frac{D}{2}}{0.5(4C - D^2)^{0.5}} \right] - \frac{1}{2C} \ln|h|$$

$$- \frac{1}{2C} \left| \frac{(K - Ct)^2}{h} - \frac{D(K - Ct)}{h^{0.5}} + C \right| - \ln B2 = 0. \quad (8)$$

We arrive at these coordinates in the form of equations (9) and (10) after some transformations:

$$hm2 = \exp\left[\frac{2D}{(4C - D^2)^{0.5}} \operatorname{arctg}\left[\frac{D}{(4C - D^2)^{0.5}} \right] - \ln C - 2C \ln B2 \right], \quad (9)$$

$$Te2 = \frac{K - D\left\{\exp\left[\frac{2D}{(4C-D^2)^{0.5}} \cdot \text{arctg}\left[\frac{D}{(4C-D^2)^{0.5}}\right] - \ln C - 2C \cdot \ln B2\right]\right\}^{0.5}}{C}. \quad (10)$$

The constants $B1$ and $B2$ that appear in equations (5)–(10) can be determined from already developed equations [9], depending upon the character of the sewage inflow and the duration of rainfall after the filling levels h_0^r, h_0^w, h_0^m have been reached for $D^2 - 4C > 0$, and levels h_1^r, h_1^w, h_1^m for $D^2 - 4C < 0$ at characteristic discrete ranges of the variation QA.

4. DISCUSSION OF NUMERICAL SOLUTIONS

Hydraulic models have been developed for multichamber reservoirs on the basis of the model designed for traditional reservoirs, in such a way as to permit theoretical experimentation and mathematical analysis of the characteristic filling and evacuation phases of sewage storage. These models have been solved numerically, using algorithms and computational programs; an extremely large number of tasks have been defined and solved over a wide range of variation in model parameters.

The results of the numerical solutions made it possible to conduct a complete analysis of the sewage accumulation process in reservoirs that relieve hydraulic conditions. Against a background of additional detailed research on sewage accumulation in traditional reservoirs, we have highlighted the distinctive and specific character of this phenomenon in the proposed physical models of multichamber reservoirs for the adopted method of determining the variation in sewage flow into a reservoir.

The resulting solutions have been evaluated on the basis of the analysis of the course of the process under examination. Theoretical importance of solutions obtained which are considered as a novelty of our research has been emphasized.

Theoretical bases and computational programs that have been developed make it possible to conduct a thorough examination of the dynamics of the changes in the filling stage in the chambers of traditional and multichamber reservoirs and to determine the curvilinear hydrograph of the outflow for any given function of sewage inflow into a reservoir.

The proposed mathematical model of a storage reservoir in a stormwater drainage system [1] has been based on the method of extreme intensities being applied in the description of variation in the inflow of stormwater sewage into a reservoir. This model can be used for designing traditional reservoirs in a stormwater sewer system.

Keeping in mind the limited utility of this model, a universal mathematical model of a traditional reservoir has been developed on the basis of a description of sewage inflow into a reservoir in the form of any given function being transformed into a series of elementary linear functions. The computational program UNO makes it possible to investigate the process of sewage accumulation for any given shape of inflow hydrograph.

The hydrograph of industrial sewage inflow given as a curvilinear function has been transformed into a series of linear functions (figure 1). In the twenty-four hour period being investigated, the function reaches three local maxima, two flow minima, and a region of zero inflow. Using the inflow hydrograph prepared and the computational program developed, it is possible to determine the dynamics of variation over time in sewage flow into a reservoir, as well as the curvilinear function of the outflow (figure 1). This function definitely determines the resulting maximum and minimum flows into a municipal wastewater system. The geometry of the reservoir can be selected so as to ensure its complete evacuation at the most appropriate moment in the process of sewage storage (figure 1).

Based on the results obtained due to solution of more than two hundred multicriterial tasks, whose number greatly exceed the range encountered in practical applications, it has been proved that the design rainfall and the required reservoir geometry are directly dependent on the flow reduction factor (figure 2). The surface layer and the filling height affect the calculated capacity of a traditional reservoir for a given set of outflow conditions.

Additional detailed research has made it possible to determine typical retention ranges, depending upon the value of the flow reduction factor, by examining the relationship between the changes in the intensity of flow into and the outflow from a ZK reservoir at different levels of reliability in reservoir operation (figure 3). Disregarding the extreme case that occurs when sewage is stored with no outflow, i.e. $QO = 0$ and $\beta = 0$, it is possible to distinguish three characteristic ranges of slow, medium, and rapid sewage accumulation, depending on the value of the coefficient β. A limiting and characteristic moment is the termination of the storage process, which occurs when $QO = QA(TM)$ for $\beta = 1.0$.

Differentiation between the retention ranges occurs at $\beta = 0.67$. Then, regardless of the values of the model parameters, the standard time TM is equal to Tp and the time necessary for designing traditional reservoirs is the same as that necessary for designing sewage system cross-sections. Rapid sewage accumulation occurs within the range of $1.0 > \beta > 0.67$, and design rainfalls last for a very short time at $TM < Tp$ and are very heavy. The range of medium sewage accumulation has been adopted for $TM > Tp$ at the flow reduction of $0.67 > \beta > 0.50$. The range of slow sewage retention for flow reduction of $0.50 > \beta > 0.0$ involves a prolonged rainfall of low intensity at $TM > Tp$.

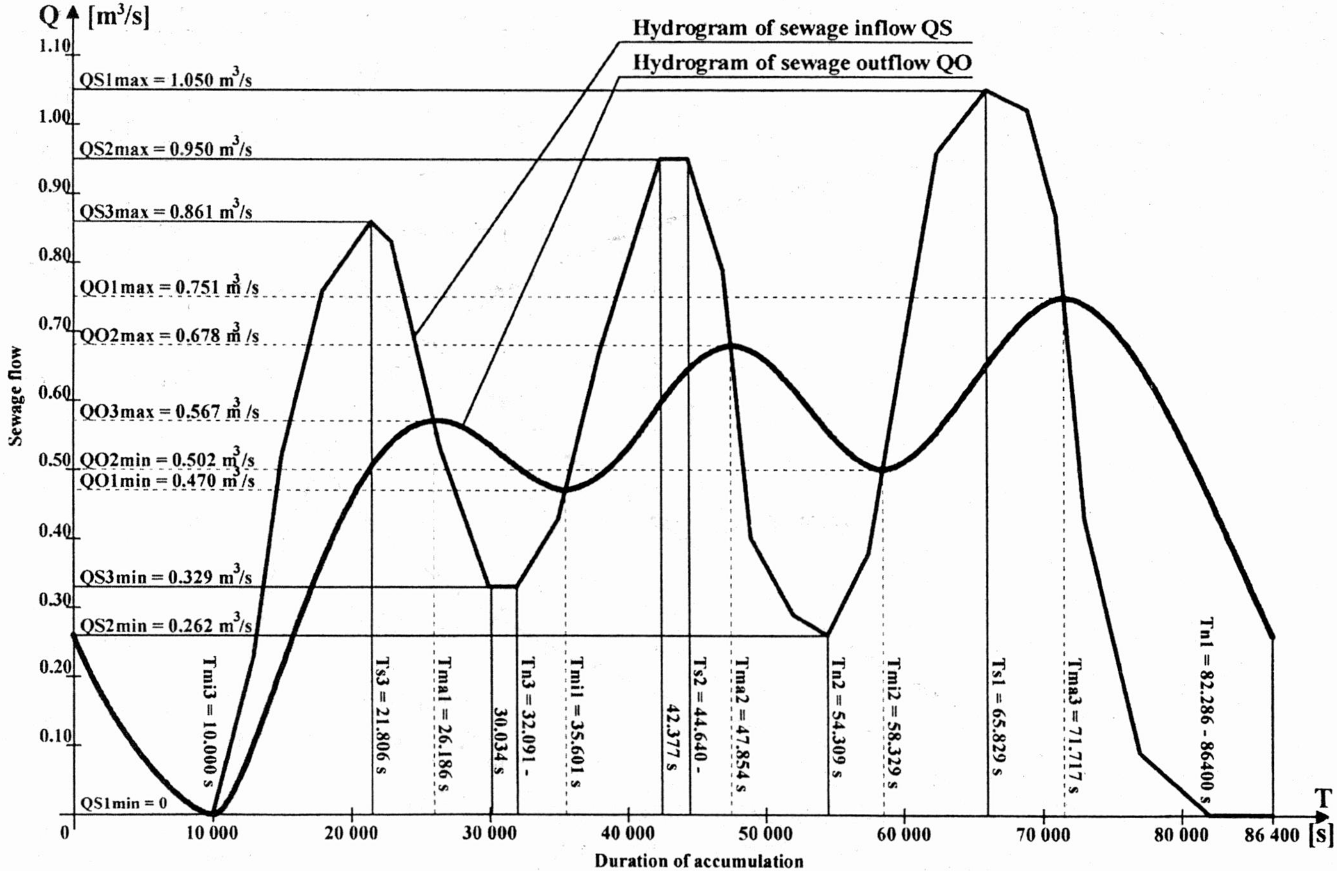

Fig. 1. Outflow hydrogram of a traditional reservoir determined by means of numerical program for a given inflow hydrogram

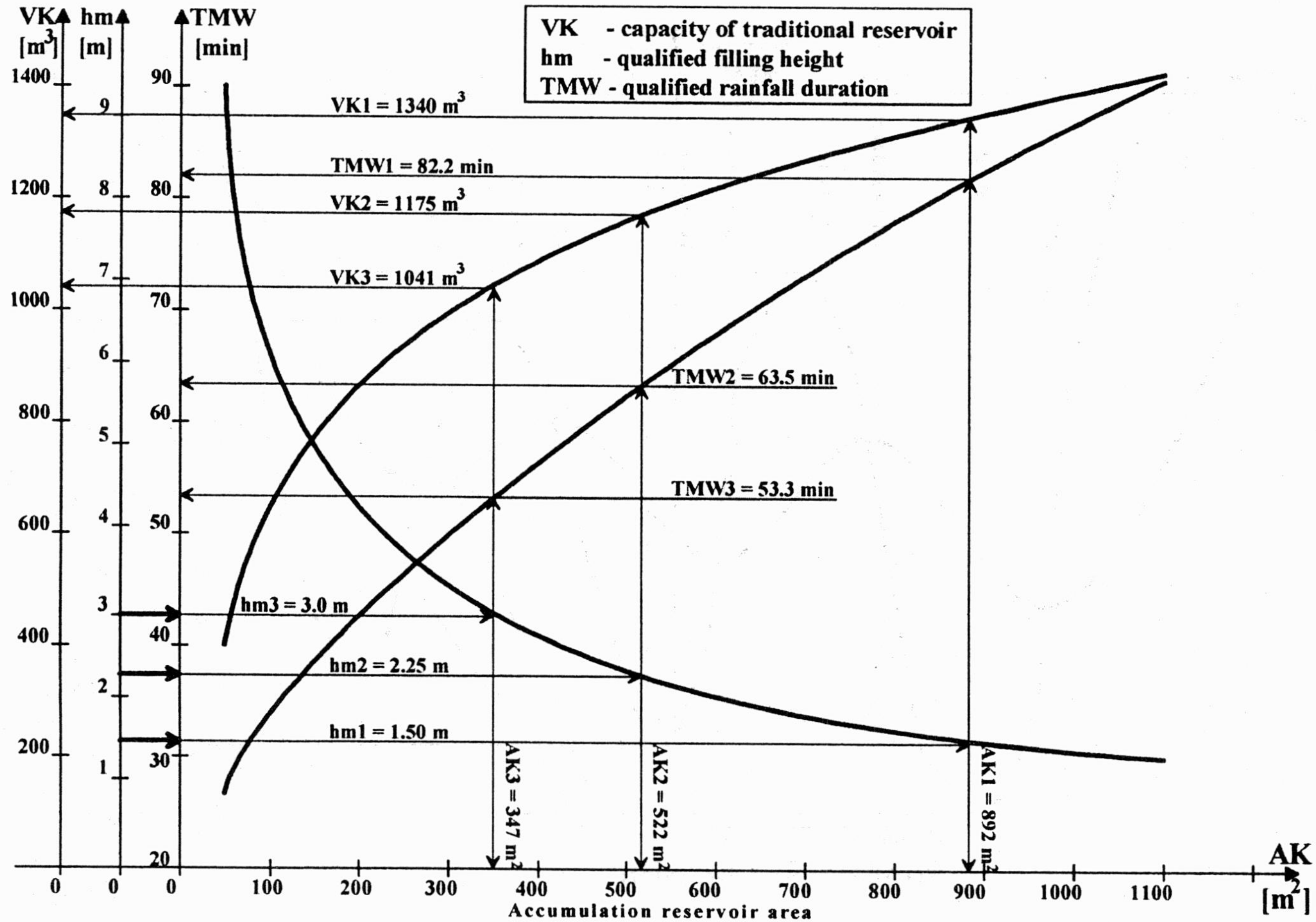

Fig. 2. Qualified rainfall duration and required reservoir capacity versus maximum filling height at various horizontal levels of traditional reservoirs

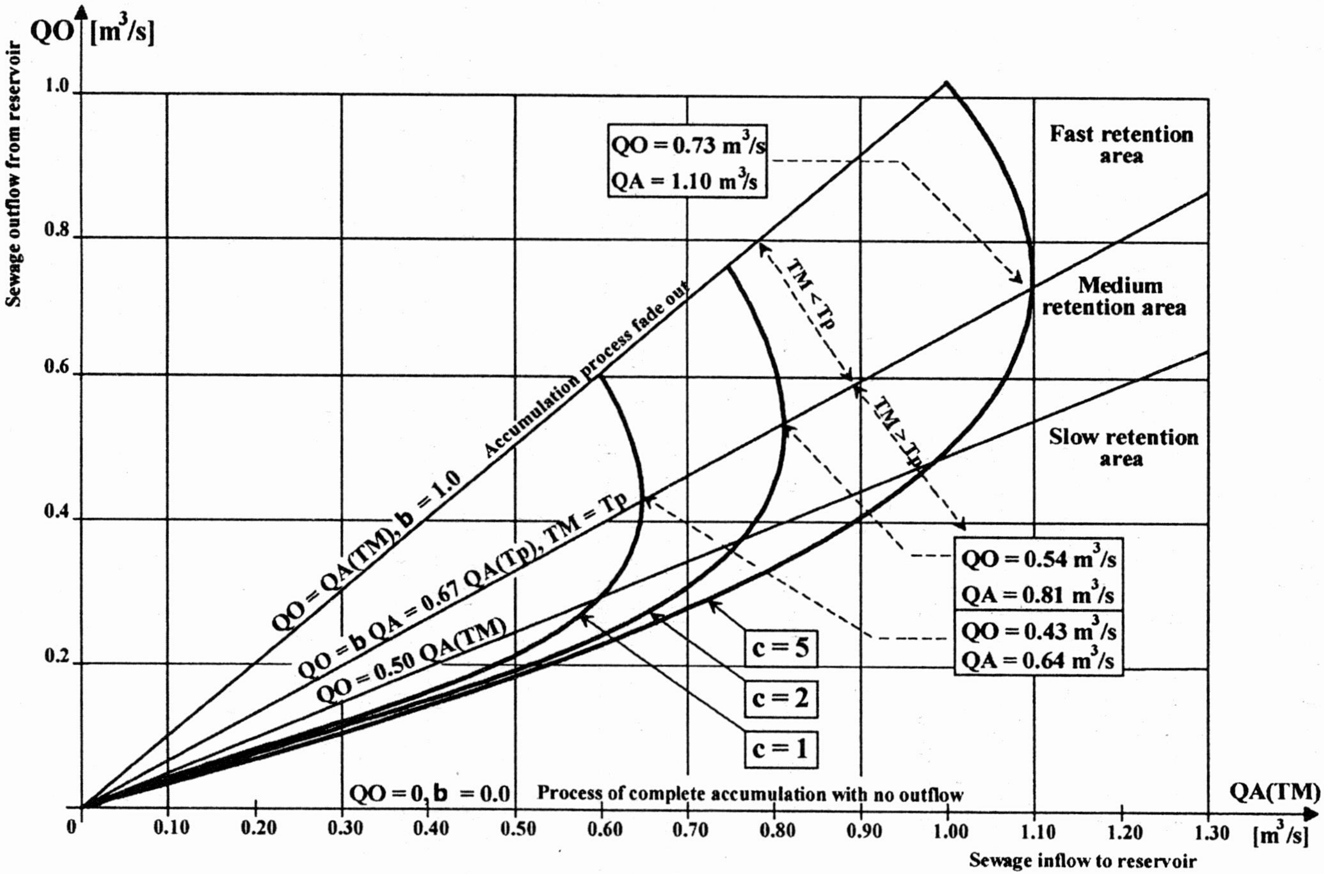

Fig. 3. Determination of traditional reservoirs' storage areas dependent on flow reduction rate

5. CALCULATING THE APPROXIMATE INDIVIDUAL CAPACITY OF TRADITIONAL RESERVOIRS

A comprehensive analysis (with the use of the program UNO) of the model parameters that characterize the sewage accumulation process in traditional reservoirs allows us to establish how the unit capacities that are necessary for the accumulation of excess sewage flowing from a reduced drainage basin $Fzr = 1.0$ ha and design rainfall TM depend on the value of the coefficient β.

Based on the analysis of the numerical solutions, quantitative relationships have been defined which makes it possible to calculate the approximate individual capacity of a traditional reservoir $VK_j(Fzr_x, \beta)$ for any given size of reduced drainage basin Fzr_r and for an assumed level of flow reduction upstream of the reservoir β

$$VK_j(Fzr_x, \beta) = Fzr^m \cdot Fzr_x^{-1} \cdot V_j(\beta) = Fzr_x^{m-1} \cdot V_j . \tag{11}$$

The capacity of a traditional reservoir can be defined as follows:

$$VK = VK_j(Fzr_x, \beta) \cdot Fzr_x . \tag{12}$$

The force value $m = 1.145$ has been adopted, and it is independent of the time of standard inflow Tp for sizing the cross-sections of the sewage system. Thus the final equation of approximate capacity of a traditional reservoir takes the following form:

$$VK = Fzr_x^{1.145} \cdot V_j . \tag{13}$$

The capacity VK of a traditional reservoir can be calculated based on formula (13) and the table. The latter has been presented in a limited range for $c = 2$, $H = 600$ mm, and $n = 0.67$. It is possible to extend considerably equation (13), considering the desired effect of the parameters c, H, and n, to the unit capacity. Then the values of the correlation coefficients α_c, α_H and α_n can be introduced and defined, hence formula (13) takes the following form:

$$VK(Fzr_x, \beta, c, H, n) = Fzr_x^{1.145} \cdot V_j \cdot \alpha_c \cdot \alpha_H \cdot \alpha_n . \tag{14}$$

This form of the formula allowing calculation of the required capacity of a traditional reservoir can have a practical value, since it reduces the complicated process of investigating sewage accumulation to a form in which the basic design parameters can be quickly determined. This form of calculation is useful when analyzing design variants and determining the economic effects of multichamber reservoirs in regulation, averaging, and control of sewage outflow in wastewater systems.

Table

Traditional reservoir unitary capacities $V_j(\beta)$ necessary for storing sewage from reduced basin $Fzr_j = 1.0$ ha and qualified rainfall duration TM depending on flow reduction rate β for $c = 2$, $H = 600$ mm and $n = 0.67$

β	TM	V_j for $Fzr = 1.0$ ha	V_{j5} for $Fzr = 5.0$ ha	V_{j20} for $Fzr = 20$ ha	V_{j50} for $Fzr = 50$ ha	V_{j100} for $Fzr = 100$ ha	V_{j500} for $Fzr = 500$ ha
	min	m^3/ha					
0.100	356.4	116.2	146.7	179.4	204.9	226.6	286.1
0.125	319.5	93.4	117.9	144.2	164.7	182.1	230.0
0.150	252.9	86.3	108.9	133.2	152.2	168.3	212.5
0.175	200.7	78.6	99.2	121.4	138.6	153.3	193.5
0.200	168.3	71.5	90.3	110.4	126.1	139.4	176.1
0.250	123.7	62.3	78.7	96.2	109.8	121.5	153.4
0.300	99.0	55.3	69.8	85.4	97.5	107.8	136.2
0.350	82.3	48.6	61.4	75.0	85.7	94.8	119.7
0.400	69.7	43.1	54.4	66.5	76.0	84.0	106.1
0.450	61.2	37.5	47.4	57.9	66.1	73.1	92.3
0.500	55.3	32.7	41.3	50.5	57.7	63.8	80.5
0.550	50.0	28.8	36.4	44.5	50.8	56.1	70.9
0.600	47.7	25.1	31.7	38.7	44.3	48.9	61.8
0.650	45.0	21.0	26.5	32.4	37.0	40.9	51.7
0.700	43.2	17.5	22.1	27.0	30.8	34.1	43.1
0.800	42.2	10.6	13.4	16.4	18.7	20.7	26.8
0.900	41.4	4.8	6.1	7.4	8.5	9.4	11.8
1.000	0.0	0.0	0.0	0.0	0.0	0.0	0.0

6. CONCLUSIONS

The method proposed makes it possible to determine in an analytical manner the values of the coordinates of the point on a curve that represents the variations in filling levels in a single-chamber reservoir at the moment when the reservoir becomes completely filled.

This method should be applied in practice, since it makes it possible to calculate the standard height of the filling level in the reservoir and the time it takes to fill the reservoir for any given rainfall duration. This calculation can practically be made for any value of the parameters that characterize the drainage basin, sewage system, geometry of the reservoir, and hydraulic capacity of the outflow channel. The method can be adapted for any sewage inflow hydrograph that is composed of a series of elementary linear functions representing variation in the sewage flow into a reservoir.

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ANALITYCZNE I NUMERYCZNE ROZWIĄZANIE RÓWNAŃ MODELI MATEMATYCZNYCH ZBIORNIKA KLASYCZNEGO

Mając na uwadze ograniczony zakres stosowania modelu matematycznego, opartego na metodzie granicznych natężeń w określaniu zmienności dopływu ścieków deszczowych do zbiornika, opracowano uniwersalny model matematyczny funkcjonowania zbiornika klasycznego. Podstawą modelu jest opis dopływu ścieków do zbiornika w formie dowolnej funkcji, przekształcanej w ciąg elementarnych funkcji liniowych. Jego praktyczne wykorzystanie z użyciem opracowanego programu obliczeniowego sprowadza się do określenia dynamiki zmian w zbiorniku klasycznym napełnionym ściekami w czasie oraz krzywoliniowej funkcji odpływu ze zbiornika. Geometrię zbiornika można dobrać tak, aby zapewnić jego całkowite opróżnienie w zadanym momencie retencjonowania ścieków. Wyniki badań teoretycznych pozwoliły także wyznaczyć charakterystyczne obszary retencji w zależności od poziomu redukcji przepływu.