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AXIOMATIC EXTENSION OF RISK MEASUREMENT

AKSJOMATYCZNE ROZSZERZENIE POMIARU RYZYKA

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Summary: In the article the author introduce the additional axiom of measure of risk and checks, mathematically proving, which well-known functions of risk fulfill this additional axiom. This will be conducted for functions such as: Value at Risk, Expected Shortfall, Median, Absolute Median Deviation, Maximum, Maximum Loss, Half Range, and Arithmetic Average. In other words, the purpose of the paper is studying which of the above functions fulfill the additional axiom of measure of risk, which can enrich Arzner's and other axioms. This axiom is not a consequence of Arzner's and other axioms. Furthermore, the author researches mathematically if the mentioned functions of risk retain their properties after replacing the partial order with the stochastic order. Finally the author presents the new measure of risk which fulfills all the axioms of measure of risk and the additional axiom.

Keywords: axioms of risk measure, coherence, VaR, ES.

Streszczenie: W artykule autor wprowadza dodatkowy aksjomat mierzenia ryzyka i sprawdza, za pomocą metod matematycznych, które z dobrze znanych funkcji ryzyka spełniają ten dodatkowy aksjomat. Dowody zostaną przeprowadzone dla takich funkcji, jak: wartość zagrożona, oczekiwany niedobór, mediana, bezwzględne odchylenie średnie, maksymalna strata, półrozstęp i średnia arytmetyczna. Innymi słowy, celem tego artykułu jest zbadanie, które funkcje ryzyka spełniają dodatkowy aksjomat miary ryzyka, który może wzbogacić aksjomatykę Arznera i innych. Ten aksjomat nie jest konsekwencją znanej aksjomatyki miary. Co więcej, autor zbada matematycznie, czy wspomniane funkcje ryzyka zachowują swoje właściwości po zastąpieniu częściowego porządku stochastycznej porządkiem częściowym. Wreszcie autor przedstawia nową miarę ryzyka, która spełnia wszystkie aksjomaty miar ryzyka i wspomniany dodatkowy aksjomat.

Słowa kluczowe: aksjomaty miary ryzyka, koherentność, VaR, oczekiwany niedobór.

1. Introduction

In this article the author will concentrate on measures of risk popular in practice and science. The author will also take into consideration some new measures of risk and check the property of monotonicity of these different risk measures for another definition of random variable order, which is the mathematically stochastic order. This will be conducted for functions such as: Value at Risk, Expected Shortfall, Median, Absolute Median Deviation, Maximum Loss, Half Range, Maximum, and Arithmetical Average. The purpose of the paper is also to study which of the above functions fulfill the additional axiom of measure of risk, which can enrich Artzner's and other axioms. This survey enables the **enlargement** of the risk measurement theory, and its new application. Another result was presented in the paper of Consigli, Kuhn and Brandimatre [Consigli et al. 2017], where the axioms of Artzner enriches dynamic context.

2. Methods

At the beginning the author presents the definition of the measure of risk.

Definition (Risk measure)

Measure of risk is a function which maps the elements of some linear subspace V of some random variables space on (Ω, F, P), which contains the constants in real variables space.

$$\rho : V \to R,$$

It fulfills the following axioms

1) monotonicity

$$\text{for every } X, Y \in V, \text{ if } X \le Y \text{ then }, \rho(X) \le \rho(Y).$$

This means that if portfolio X generates losses with a smaller probability, then the risk joined with this portfolio is smaller.

2) invariance: for every $a \in R$ and for every $X \in V$

$$\rho(X + a) = \rho(X) + a.$$

This axiom may be interpreted that when we add some money to the portfolio with value the risk joined with this portfolio is rising, because we may invest more money and lose more. As the values of risk measures are real we can compare them and order them if they fulfill the above axioms [Artzner et al. 1997].

Definition (Coherent measure)

The measure of risk is coherent if it fulfills the conditions:

1) positive homogeneousness

for every $\lambda \geq 0$ and for every $X \in V$ the truth is that

$$\rho(\lambda X) = \lambda\rho(X).$$

This axiom may denote that multiplying the quantity of investment causes the risk to increase proportionally. An example may be the leverage effect in stock market investing.

2) subadditivity

$\forall\ X, Y \in V$ there exists the relation:

$$\rho(X+Y) \leq \rho(X) + \rho(Y).$$

In a well-diversified portfolio the total risk of a loss value is not bigger than the risk of its individual loss values. The rules of coherence allow for the consequence in risk assessment [Artzner et al. 1997; Uniejewski 2004]. We will sum up the information about risk measures researched in this article. The Value of Risk is the biggest value that can be lost as a result of investing in a portfolio with a given time horizon and with a given tolerance level [Best, Komański 2000]. *VaR* is defined as a loss which cannot be overrun or achieved. It is very popular and universal and used by banks, investment funds, pension funds, and investment houses. There exists some modifications of this measure, RiskMetrics, CFaR, EaR. There exist two alternative models of VaR. Jajuga [2007] defined VaR as a special quantile

$$P(W > W_0 - VaR) = 1 - \alpha$$

where: W – a market value at the end of the considered period, W_0 – a market value in a given moment α – a tolerance level. In this article we will combine it with another written definition of VaR

$$\inf\{x \in R : F(x) \geq \alpha\}.$$

On the basis of VaR Expected Shortfall was created, also called the conditional value of risk, and denoted as CVaR or TVAR. ES assesses the value of risk in a classical way focusing on the external results. It is clear that the expected loss on the portfolio may be equal or higher than some quantile. Usually one assumes for the calculations of *ES* – 5% level of confidence. Formally *Expected Shortfall* may be defined as follows:

$$ES_\alpha = \frac{1}{\alpha}\int_0^\alpha VaR_\gamma(X)d\gamma,$$

and in a discrete case as follows:

$$ES_\alpha = -\frac{1}{\alpha}\left(E\left[X1_{\{X \leq x_\alpha\}} + x_\alpha\left(\alpha - P\left[X \leq x_\alpha\right]\right)\right]\right).$$

Expected Shortfall see [Trzpiot 2004; Acerbi, Tasche 2002], may be interpreted as the mean of the worst $(1-\alpha)$ % losses on condition that these losses are bigger than the value of risk. The other measures considered in this article are plain and do not demand comment, such us *ML*, Maximum, Median, Median Absolute Deviation in a continuous and discrete case. In this article we will take into consideration two alternative definitions of random variables.

Definition (Standard definition of stochastic order of n degree)

If the variable X dominates stochastically the variable *Y*, which can be written $X \le Y$, then

For $n = 1$

$$F_1(y) \le F_2(x)$$

$$\int_{-\infty}^{y} F_1^{\,n-1}(t)dt \le \int_{-\infty}^{x} F_2^{\,n-1}(t)dt.$$

This means that with variable *Y* there is a bigger risk then the risk with variable *X*. As some measures of risk do not include probability we can define the order of relations without considering probability.

Weak order, partial order is the reflexive, transitive and antisymmetric relation. In this way we will define the relation of the order on stochastic variables.

Definition (Partial order on random variables)

Assume that:

$$X = (x_1, x_2, ..., x_k), \; Y = (y_1, y_2, ..., y_k)$$

$$X \le Y \Leftrightarrow \underset{x_j, y_j}{\forall} x_j \le y_j .$$

In this paper the author will analyze mathematically which of the functions of measure are monotonic with this definition of order on random variables. The author will check which of the risk measures fulfill the additional axiom based on the paper which was yet not published, by the same author. *X* and *Y* are risk variables. They may have two different interpretations as the value of the portfolio and also the value of its part. The axiom is the following:

Axiom

From any risk variables called X and Y

$$X \le Y \Rightarrow \mu(Y / X) = \mu(Y) - \mu(X)$$

This axiom may be interpreted that if the portfolio variable *X* is smaller than the bigger portfolio variable *Y*, then the risk combined with variable *Y* after removing variable X is equal to the risk of Y minus the risk of X. The author claims that the above axiom is not an obvious consequence of the axioms of risk measure and coherence. It is a modification of one of the basic characteristics of measure resulting from the axioms of mathematical measure. Thanks to this new axiom it is possible to accurately calculate the risk of the difference of two random variables. This axiom is stronger than the axiom of sub-additivity and is not an obvious consequence of the axioms of sub-additivity and homogeneousness. The risk of difference may be helpful when we reduce the portfolio. For example a bank or an insurance company in the portfolio of loans or insurance policies eliminates the risk through securitization or reinsurance. These companies are interested in the risk assessment of the investment portfolio after such a reduction.

This axiom has one meaning. The left side may be only a partial order. For stochastic order the author suggests the following axiom

$$Y \leq X \Rightarrow \mu(Y / X) = \mu(Y) - \mu(X).$$

If the probability of crossing the loss limit is higher for portfolio variables Y than for variables X, then the risk of the portfolio after reduction should be calculated as risk Y minus risk of X. We will prove which of the measures of risk fulfill this axiom.

In the counterexamples below, the author uses the following formula for the probability of a difference:

$$P(X - Y) = P(X) - P(X \cap Y).$$

3. Mathematical proof

First the author analyzes the most popular function of risk.

$$VaR = \inf\{x : F(x) \geq \alpha\}$$

Counter-example

Let us define the probability distribution of a random variable. Suppose that $X \leq Y$

Table 1. The distribution of random variable

x_i	**4**	**7**	x_i	2	3
p_{yi}	**1/2**	**1/2**	p_{xi}	1/2	1/2
$y_i - x_j$	*1*	*2*	*4*	*5*	
p_{ij}	*1/4*	*1/4*	¼	*1/4*	

Source: own example.

$$\mu(Y / X) = \inf\left\{y_i - x_j : F\left(y_i - x_j\right) \geq 0{,}5\right\} = 4$$

$$\inf\{y_i : F(y_i) \geq 0,5\} - \inf\{x_i : F(x_i) \geq 0,5\} = 4 - 2 = 2.$$

So $\rho(Y/X) \neq \rho(Y) - \rho(X)$.

Secondly, the author considers the expected value

- Expected Value

Counter-example

We notice that $X \leq Y$

$$\sum_{i=1}^{n}(x_i - y_j)p_{ij} = \sum_{i=1}^{n} x_i p_{ij} - \sum_{i=1}^{n} y_j p_{ij}, \ \ \rho(Y/X) = \rho(Y) - \rho(X).$$

Thus the expected value fulfills the additional axiom.

- Absolute Median Deviation

Counter-example.

Table 2. The distribution of random variable

y_i	**4**	**5**	x_i	2	3
p_{y_i}	**1/2**	**1/2**	p_{x_i}	1/2	1/2
$y_i - x_i$			*1*	*2*	*3*
p_i			*1/4*	*2/4*	*1/4*

Source: own example.

$$Med\left(\left|Y - X - Med(Y-X)\right|\right) = Med\left(\left|\{5-3-2, 5-2-2, 4-3-2, 4-2-2\}\right|\right)$$
$$= Med\left(\{|1-2|, |2-2|, |3-2|\}\right) = 0,5$$

$$Med(Y - Med(Y)) = Med\left|\{5-4,5; 4-4,5\}\right| = Med\left(\{|-0,5|, |0,5|\}\right) = 0,5$$

$$Med(X - Med(X)) = Med\left(\{|2-2,5|, |3-2,5|\}\right) = 0,5$$

$$0,5 > 0 \Rightarrow \rho(Y/X) \neq \rho(Y) - \rho(X).$$

In the paper entitled “About the fundamentals of measures of risk”, the author proved that Median is a coherent measure of risk when we define a sum of random variables in a particular way. That is way it is presented in this survey.

- Median

Counter-example

Table 3. The distribution of random variable

y_i	**4**	**6**		x_i	2	3	3,5
p_{yi}	**1/2**	**1/2**		p_{xi}	1/3	1/3	1/3
$y_i - x_j$		*0,5*	*1*	*2*	*2,5*	*3*	*4*

Source: own example.

$$Med(Y - X) = 2{,}25 \;\; Med(Y) = 5 \;\; Med(X) = 3 \, .$$

So

$$\rho(Y / X) \neq \rho(Y) - \rho(X).$$

The median does not fulfill the additional axiom.

- Maximum
  Counter-example.

Table 4. The distribution of random variable

y_i	**4**	**5**	x_i	2	3
p_{yi}	**1/2**	**1/2**	p_{xi}	1/2	1/2

Source: own example.

$$\max(5-3,5-2,4-3,4-2) = \max(1,2,3) = 3$$
$$\max(5,4) = 5 \;\; , \max(3,2) = 3$$
$$5 - 3 = 2 \Rightarrow \rho(Y / X) \neq \rho(Y) - \rho(X).$$

We will analyze ML as a well- known measure of risk, presented for example in [Czerniak, 2003].

- $ML = \max_i \, p_i x_i$
  Counter-example *.

Table 5. The distribution of random variable

y_i	**4**	**7**	x_i	2	3
p_{yi}	**1/2**	**1/2**	p_{xi}	1/2	1/2
$y_i - x_j$		*1*	*2*	*4*	*5*
p_{ij}		*1/4*	*1/4*	*1/4*	*1/4*

Source: own example.

$$\rho(Y - X) = \max_{ij} \, p_{ij}(y_i - x_j) = \frac{5}{4}.$$
$$\rho(Y) - \rho(X) = \max_i \, p_{y_i} y_i - \max p_{xj} x_j = \frac{7}{2} - \frac{3}{2} = 2.$$

So $\rho(Y-X) \neq \rho(Y)-\rho(X)$.

Alternatively from the condition of independence of random variables:

$$\rho(Y-X)=\max_{ij} p_{ij}(y_i-x_j)=\max_{ij} p_i p_j(y_i-x_j)=\max_{ij}(p_i p_j y_i - p_i p_j x_j) \leq$$
$$\max_{ij}(p_i y_i - p_j x_j)=\max_i p_i y_i - \max p_j x_j = \rho(Y)-\rho(X).$$

- Half Range

  Counter-example.

  Let us take into account the counter-example*

$$\rho(Y-X)=0{,}5(5\text{-}1)\text{=}2\,,\quad \rho(Y)-\rho(X)=0{,}5(7-4)-0{,}5(3-2)=\frac{3}{2}-\frac{1}{2}=1\,.$$

So

$$\rho(Y-X) \neq \rho(Y)-\rho(X).$$

Finally, the author will check which of the above functions of risk are still monotonic after changing the stochastic order on partial order

$$VaR=\inf\{x : F(x) \geq \alpha\}$$

Assume that $X \leq Y \Rightarrow \forall_j\, x_j \leq y_j$.

So

$$\inf\{x : F(x) \geq \alpha\} \leq \inf\{y : F(y) \geq \alpha\} \Rightarrow \rho(X) \leq \rho(Y).$$

- Expected Value

  Assume that $X \leq Y \Rightarrow \forall_j\, x_j \leq y_j$.

  Counter-example

Table 6. The distribution of random variable

x_i	**1**	**2**	**3**	y_i	1	4	4,5
p_{xi}	**1/3**	**1/3**	**1/3**	p_{yi}	0,8	0,1	0,1

Source: own example.

$$\sum_i x_i p_{xi} = 2\ ,\sum_i y_i p_{y_i} = 1{,}65 \Rightarrow \rho(X) > \rho(Y).$$

- Absolute Median Deviation

  Assume that

$$X \leq Y \Rightarrow \forall_j\, x_j \leq y_j\ .$$

Counter-example

Table 7. The distribution of random variable

y_i	**1**	**1,2**	**1,3**	x_i	0	0,5	1
p_{yi}	**1/3**	**1/3**	**1/3**	p_{xi}	1/3	1/3	1/3

Source: own example.

$$Med\left|X - Med(X)\right| = 0{,}5, \qquad Med\left|Y - Med(Y)\right| = 0{,}1.$$

So

$$Med\left|X - Med(X)\right| > Med\left|Y - Med(Y)\right| \text{ So } \rho(X) > \rho(Y).$$

- Median(X)

$$X \le Y \Rightarrow \underset{j}{\forall} x_j \le y_j \Rightarrow Med(X) \le Med(Y). \text{ So } \rho(X) \le \rho(Y).$$

- Maximum

$$X \le Y \Rightarrow \underset{j}{\forall} x_j \le y_j \Rightarrow Max(X) \le Max(Y). \text{ So } \rho(X) \le \rho(Y).$$

- $ML = \max_i p_i x_i$

  Counter-example

$$X \le Y \Rightarrow \underset{j}{\forall} x_j \le y_j \Rightarrow \max_i p_{xi} x_i = 1, \ \max_i p_{y_i} y_i = 0{,}8.$$

Table 8. The distribution of random variable

x_i	**1**	**2**	**3**	y_i	1	4	4,5
p_{xi}	**1/3**	**1/3**	**1/3**	p_{yi}	0,8	0,1	0,1

Source: own example.

$$\text{So } \rho(Y) < \rho(X).$$

The author conducts the proof for Half Range which is not a coherent measure of risk.

- Half Range

  Counter-example

Table 9. The distribution of random variable

x_i	**1**	**2**	**3**	y_i	3	4	4,5
p_{xi}	**1/3**	**1/3**	**1/3**	p_{y_i}	0,8	0,1	0,1

Source: own example.

$$X \le Y \Rightarrow \underset{j}{\forall} x_j \le y_j \Rightarrow 0{,}5\left(X_{\max} - X_{\min}\right) = 1,\ 0{,}5\,(Y_{\max} - Y_{\min}) = 0{,}75\,.$$

So

$$\rho(Y) < \rho(X).$$

- Arithmetic Average

$$X \le Y \Rightarrow \underset{j}{\forall} x_j \le y_j \Rightarrow \frac{1}{n}\sum_i x_i \le \frac{1}{m}\sum_j y_j \Rightarrow \rho(X) \le \rho(Y)\,.$$

The author will prove that Arithmetic Average fulfills the axioms of coherent measure of risk and the additional axiom

- Arithmetic Mean

$$\rho(Y - X) = \frac{1}{nm}\sum_i\sum_j\left(y_i - x_j\right) == \frac{1}{n}\frac{1}{m}\sum_i\sum_j y_i - \frac{1}{n}\frac{1}{m}\sum_i\sum_j x_j = \frac{1}{n}\sum_i y_i - \frac{1}{m}\sum_j x_j = \rho(Y) - \rho(X).$$

The author will prove that the Arithmetic Mean is a coherent measure of risk

Monotonicity

$$X \le Y \Rightarrow F_1(y) \le F_2(x)\,.$$

So

$$\frac{1}{n}\sum_i x_i \le \frac{1}{n}\sum_j y_j \Rightarrow \rho(X) \le \rho(Y).$$

Homogenousness

From the properties of the number series the author concludes:

$$\rho(\lambda X) = \frac{1}{n}\sum_i \lambda x_i = \lambda\frac{1}{n}\sum_i x_i = \lambda\rho(X)\cdot$$

Strong sub-additivity

From the properties of number series the author concludes:

$$\rho(X + Y) = \frac{1}{n}\frac{1}{m}\sum_i\sum_j\left(y_i + x_j\right) = \frac{1}{n}\frac{1}{m}\sum_i\sum_j y_i + \frac{1}{n}\frac{1}{m}\sum_i\sum_j x_i = \frac{1}{n}\sum_i y_i + \frac{1}{m}\sum_j x_i = \rho(Y) + \rho(X).$$

Invariance

From the properties of number series the author concludes:

$$\rho(X+a)=\frac{1}{n}\sum_{i=1}^{n}(x_i+a)=\frac{1}{n}\sum_{i=1}^{n}x_i+\frac{1}{n}\sum_{i=1}^{n}a=\frac{1}{n}\sum_{i=1}^{n}x_i+a=\rho(X)+a$$

4. Conclusions

In this survey the author proved that functions of risk such like VaR, Median Absolute Deviation, Median, Maximum , Maximum Loss and Half Range do not fulfill the additional axiom of measure of risk. When the author takes into consideration the partial order as the order on random variables it occurs that VaR, Median, Maximum and Arithmetic Mean are monotonic and E(X), Absolute Median Deviation, Maximum Loss and Half Range are not monotonic. The example of coherent measures of risk which is monotonic with partial order and fulfills the additional axiom also, is the Arithmetic Average. Unfortunately it does not include probability.

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