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A STACKELBERG GAME IN A PRODUCTION-DISTRIBUTION SYSTEM WITH MULTIPLE BUYERS[1]

This paper investigates the coordination of deliveries between a vendor (or manufacturer) and multiple heterogeneous buyers (or retailers) in a two-level supply chain with a decentralized decision process. A continuous deterministic model is presented. To satisfy the buyers' demands, the vendor delivers the product in JIT shipments to each buyer. The buyers' demands (continuous) have to be satisfied by the vendor. The production rate is constant and sufficient to meet the buyers' demands. The product is delivered in discrete batches from the vendor's stock to the buyers' stocks and all shipments are realized instantaneously. A special class of production-delivery-replenishment policies of the vendor and the buyers are analyzed. In a competitive situation, the objective is to determine schedules, which minimize the individual average total cost of production, shipment and stockholding in the production-distribution cycle (PDC).

This paper presents a game theoretic model without prices, where agents minimize their own costs. It is a non-cooperative $(1 + n)$-person constrained game with agents (a single vendor and n buyers) choosing the number and sizes of deliveries. The model describes inventory patterns and the cost structure of PDC. It is proven that there exist equilibrium strategies in the considered Stackelberg sub-games with the vendor as the leader. Solution procedures are developed to find the Stackelberg game equilibrium.

Keywords: *supply chain, constrained game, Stackelberg game*

1. Introduction

In general, a supply chain is composed of independent partners with individual costs. For this reason, each firm is interested in minimizing its own costs independ-

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ently. Both in practice and in the literature, considerable attention is paid to the importance of a coordinated relationship between distinct entities (as supplier, manufacturer, transporter, buyer etc.) in the supply chain. Supply chain management has recently received a great deal of attention in economics. The idea of joint optimization for the vendor and buyer was initiated by GOYAL [8], BANERJEE [2], and LU [11]. A basic policy is any feasible policy where deliveries are made only when a buyer has zero inventory. Several authors (see the literature review in [12]) incorporated policies in which the sizes of successive deliveries from the vendor to a buyer within a production cycle either increase by some factor or are equal in size. For the one buyer case, Hill, [9], shows that in an optimal cycle the total production is transferred in deliveries of initially increasing and then equal size (see Fig. 1). Some researchers, for example in [3], [6], [10] and [13], suggest quantitative models to describe the motivation and negotiating tools for providing joint operating policies. The effects of incorporating transportation costs into the model on the possibility of better decision making have also been studied. In most papers dealing with integrated inventory models, for example [7], transportation costs are considered only as part of the fixed replenishment cost. A proposal for partitioning delivery and transportation costs for deliveries differing in size is given in [5] for the non-integrated case.

However, there is an additional set of problems involved in implementing policies (strategies) with respect to whether and how the agents participate in the delivery-transportation costs in the case of multiple buyers. Most authors (we refer the reader to [11]–[14] and [16]) proposed models in which deliveries to a given buyer are identical in size. The replenishment instants are specified either by the vendor (the vendor's stock is empty immediately after replenishment) or by the buyers (in the JIT mode for the buyer), see ZAVANELLA and ZANONI [16] for an analytic formulation of consignment stock policies and the references given there.

This paper presents a game-theoretic approach for the case with non-equal sizes of deliveries. Most game-theoretic models of the supply chain assume agents maximize their individual profit functions (with respect to purchase and sale prices), see [15] and its comprehensive literature review. A model where agents minimize their individual costs under the assumption that only the division of shipment costs is coordinated centrally or negotiated was studied in [4] and [5]. The research presented in this paper presents a model of a Stackelberg game without prices, where agents minimize their individual costs. It is a non-cooperative game theoretic model of single-vendor and multi-buyer competition in terms of the number and sizes of deliveries to the buyers. Generalizations of some results of [5] with respect to multiple buyers are given. The main results of this paper were presented in [4].

The paper is organized as follows. In Section 2, we develop the model describing inventory patterns and the cost structure under a production-distribution cycle (PDC). It is then assumed that the agents (the vendor, as a leader, and the buyers) compete over batch sizes in sub-games. The existence of equilibrium multi-strategies is proved in Sections 3 and 4.

2. Vendor–buyer relationships under a production-distribution cycle (PDC)

We consider a continuous deterministic model of a production-distribution system for a single good. The players are denoted by the indices $i = 0$ for the vendor and $i = 1, ..., n$ for the buyers, respectively. The production function and buyers' demands are linear functions of time. The vendor produces a good and supplies it to the buyers in discrete deliveries (batches) as in [2] and [7]. Buyers are not homogenous – their demand rates and cost parameters can differ.

2.1. Description of the model and its assumptions

Specifically, the problem is characterized by the following conditions:

1. *The production rate* $P > 0$ *and individual demand rates* $D_1 > 0, ..., D_n > 0$ *are constant over time.*
2. *The production rate P is sufficient to meet total demand, i.e.*

$$\lambda = \frac{P}{D} > 1, \quad \textit{where} \quad D = D_1 + ... + D_n.$$

3. *The final product is distributed by shipping it in discrete batches from the vendor's stock to the buyers' stocks (realized instantaneously).*
4. *Each buyer receives shipment just as it runs out of stock (replenishments only if inventory positions are zero).*
5. *There are the following cost parameters:*
   A = fixed production set up cost,
   A_i *= fixed ordering/shipment cost for* $i = 0, 1, ..., n$,
   h_i *= unit stock holding costs, for* $i = 0$.

In the case of central coordination, the problem is to find a schedule which minimizes the average total (production, shipping, replenishment and holding) common cost for a given (or infinite) time horizon. Production-distribution schedules for long time horizons are sequences of production-distribution cycles (PDC). For an infinite time horizon, an optimal schedule contains only cycles with minimal average costs – called economic production-distribution cycles (EPDC). In this paper we investigate a case without central coordination with additional assumptions:

6. *There is exactly one production setup for a total production of size Q at the beginning of the production-distribution cycle* (*i.e.* $t^* = \frac{Q}{P}$ *is the production time in the PDC and* $T = \frac{Q}{D}$ *is the length of the PDC*).

7. *In a schedule of shipments in PDC, for each i* = 1, ..., *n*,
buyer i receives $k_0 + k_i > 0$ *deliveries such that*
a) *the vendor's stock becomes empty at the times of the* $k_0 \geq 0$ *initial deliveries and*
b) *the last* $k_i > 0$ *deliveries are equal in size.*

Additionally, we write $I_i(t)$ for the inventory positions (just before possible replenishment).

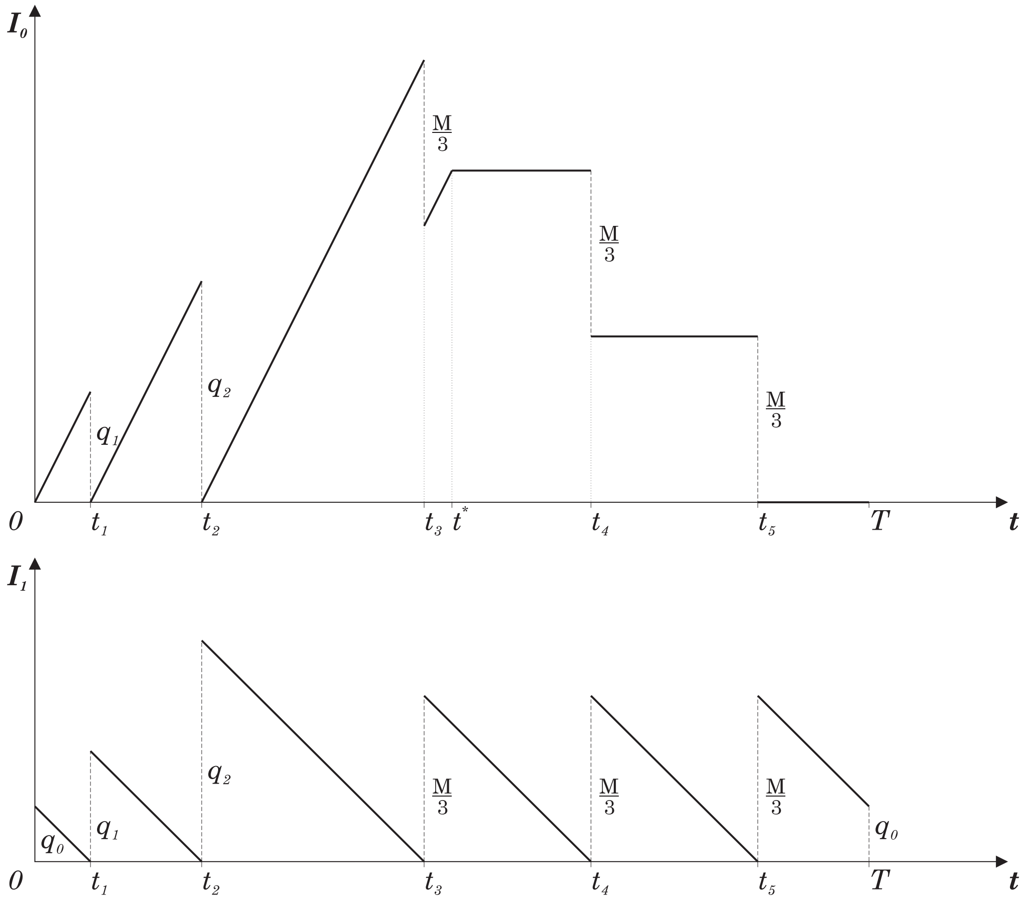

Fig. 1. The inventory positions for vendor and buyer stocks in PDC (*n* = 1)

We assume for PDC that $I_i(t) \geq 0$ *for every* $t \in [0, T]$ *and the initial inventories are the same as the final* (*i.e.* $I_i(0) = I_i(T)$ *for each* $i = 0, 1, ..., n$).

8. *The vendor controls the* k_0 *initial shipments, but buyer i controls its own* k_i *last shipments. This means that Assumption 7 holds and the appropriate transportation costs are equal to* k_0A_0 *and* k_iA_i*, respectively.*

Assumptions 1–5 are reasonable, based on practice and the majority of integrated models in the literature (see presentation in [12]).

Example 1. *A schedule satisfying postulates* 1–8 (*for the case of* $n = 1$, $k_0 = 2$, $k_1 = 3$) *is presented in Fig. 1. The buyer receives each shipment just as it runs out of stock. The vendor's stock becomes empty just after each of 2 increasing deliveries* q_1 *and* q_2*, but the last 3 deliveries are equal in size. The average cost of the schedule has two components*

$$\begin{cases} \dfrac{1}{T}\left[A + 2A_0 + h_0\displaystyle\int_0^T I_0(t)dt\right] & \text{for the vendor,} \\ \dfrac{1}{T}\left[3A_1 + h_1\displaystyle\int_0^T I_1(t)dt\right] & \text{for the buyer.} \end{cases}$$

For one buyer and the model with Assumptions 1–6 and $h_0 < h_1$, HILL [9] proved that the economic production-distribution cycle (EPDC) in the centralized (cooperative) case satisfies Assumptions 6–7. Schedules satisfying Assumption 7 were investigated as agents' strategies in a non-cooperative production-distribution game. Equilibrium economic production-distribution cycles (EEPDC) were described in [5].

Remark 1. *Consider a PDC described by Assumptions* 6–7. *In the PDC schedule*, *the total production is partitioned as* $Q = Q_0 + M_1 + ... + M_n$ (*with* $Q_0 = 0$ *if and only if* $k_0 = 0$) *so that* $Q_0 = \sum_{j=1}^{k_0} q_j$ *and* q_j *is the total size of deliveries to the n individual buyers in vendor mode* (*with respect to Assumptions* 7a *and* 8) *and the vendor's shipment costs are* k_0A_0. *The remaining batches of total size* $M = M_1 + ... + M_n$ *are transported so that* M_i *units are transported in individual mode* (*with respect to Assumptions* 7b *and* 8) *to buyer i in* k_i *deliveries of equal size and the costs of the i-th buyer are* k_iA_i. *The sequence* $((Q, k_0), (M_1, k_1), ..., (M_n, k_n))$ *will be called the strategic characteristic of PDC.*

To make Assumptions 1–8 precise, we use standard mathematical notation with R_+ and N being the sets of positive reals and natural numbers, respectively. The strategic characteristic of a PDC in the model considered satisfies the following:

$$((Q, k_0), (M_1, k_1), ..., (M_n, k_n)) \in (R_+ \times N)^{n+1}$$

and

$$\text{if } k_0 = 0, \quad \text{then} \quad M_i = \frac{D_i}{D} Q. \tag{1}$$

Every sequence $\pi = (\pi_0, \pi_1, ..., \pi_n) = ((Q, k_0), (M_1, k_1), ..., (M_n, k_n))$ satisfying (1), will be called a multi-strategy. We define π to be feasible if it can be obtained from a PDC (as in Remark 1). The set of all feasible multi-strategies will be denoted by $\widetilde{\Pi}$.

Remark 2. *A multi-strategy*

$$\pi = (\pi_0, \pi_1, ..., \pi_n) = ((Q, k_0), (M_1, k_1), ..., (M_n, k_n)) \in \widetilde{\Pi}$$

can be viewed as the set of the strategies of individual agents:

1. *the vendor's strategy* $\pi_0 = (Q, k_0) \in R_+ \times N$ *which determines total production* Q *and the number* k_0 *of batches in vendor mode* (*to individual buyers'*) *with the j-th batch being of size* q_j, *for* $j = 1, ..., k_0$, *if only* $k_0 > 0$,

2. *buyer i's strategy* $\pi_i = (M_i, k_i) \in R_+ \times N$, *which determines the number of deliveries* k_i *in vendor mode and their size* $q_{i,j} = \frac{M_i}{k_i}$ *for* $j = 1, ... k_i$.

A multi-strategy is feasible if it can be represented as the strategic characteristic of a PDC. Fulfilling condition (1) is not sufficient. Appropriate conditions for the feasibility of π *will now be derived.*

2.2. Feasible multi-strategies – analytical consideration

We introduce the notation: $(P, D_1, ..., D_n) \in R_+^{n+1}$ and $(A, A_0, A_1, ..., A_n, h_0, h_1, ..., h_n)$ are the technological and cost parameters in the model, respectively.

$Q \in R_+, \widetilde{M} = (M_1, ..., M_n) \in R_+^n$ and $\widetilde{k} = (k_0, k_1, ..., k_n) \in N^{n+1}$ are the decision variables.

$I_i(t)$ = the inventory position (before a possible replenishment, if any) at time $t \in [0, T]$.

$q_{i,0} = I_i(0)$ = buyer i's initial inventory position.

$q_{i,1}, ..., q_{i,k_0}$ = the sizes of the k_0 initial deliveries to buyer i shipped successively at times $t_1, ..., t_{k_0}$. We have $q_j = \sum_{i=1}^{n} q_{i,j} = \lambda^j q_0$ for each $j = 0, 1, ..., k_0$.

$q_{i,k_0+1}, ..., q_{i,k_0+k_i}$ = the sizes of the remaining deliveries to buyer i, shipped successively at times $t_{i,1}, ..., t_{i,k_i}$, respectively. We have $q_{i,j} = \frac{M_i}{k_i}$ for $j = 1, ..., k_i$.

Additionally, if $k_0 \geq 1$, by Assumptions 6–7, for $j = 0, 1, ..., k_0 - 1$, we have

$$q_{i,j} = \frac{D_i}{D} q_j = \lambda^j q_{i,0} \quad \text{and} \quad q_{i,k_0} + M_i = (T - t_{k_0}) D_i + q_{i,0}\,. \tag{2}$$

The agents' costs associated with the multi-strategy

$$\pi = (\pi_0, \pi_1, ..., \pi_n) = ((Q, k_0), (M_1, k_1), ..., (M_n, k_n))$$

are defined in a natural way:

$$V_i = \begin{cases} \dfrac{1}{T}\left[A + k_0 A_0 + h_0 \displaystyle\int_0^T I_0(t)dt \right] & \text{for the vendor,} \\ \dfrac{1}{T}\left[k_i A_i + h_i \displaystyle\int_0^T I_i(t)dt \right] & \text{for the buyer } i. \end{cases} \tag{3}$$

where each $I_i(t)$ for $t \in [0, T]$ depends on the PDC schedule (the decision parameters) and on the technological parameters.

Proposition 1. *A multi-strategy* $\pi = ((Q, k_0), (M_1, k_1), ..., (M_n, k_n))$ *is feasible if and only if there exists a collection of* $q_{i,0} > 0$, $i = 1, ..., n$, *such that the schedule* $[q_{i,j}]$ *with*

$$q_{i,j} = \begin{cases} \lambda^j q_{i,0} & \text{for } 0 < j < k_0, \\ \lambda^{k_0} q_{i,0} + \Delta_i & \text{for } j = k_0, \qquad \text{where } \Delta_i = \dfrac{D_i}{D} M - M_i, \\ \dfrac{M_i}{k_i} & \text{for } j > k_0, \end{cases} \tag{4}$$

is a PDC schedule.

Additionally, *if* $k_0 > 0$, *then there is only one such schedule*, *denoted by* $q(\pi)$, *with*

$$q_{i,0}(\pi) = \frac{D_i}{D} q_0(\pi) \quad \text{and} \quad q_0(\pi) = \frac{Q - M}{a_{k_0}}, \quad \text{where} \quad a_v = \frac{\lambda(\lambda^v - 1)}{\lambda - 1}. \tag{5}$$

Proof. If $k_0 = 0$, then $\Delta_i = 0$ for each $i = 1, ..., n$. There is the possibility of more than one initial inventory position $q_{i,0}$ and the theorem follows.

In the schedule $[q_{i,j}]$, the appropriate replenishment instants are the following:

$$\begin{aligned} & t_0 = 0 \\ & \begin{cases} t_j = t_{j-1} + \lambda^{j-1} \dfrac{q_0}{D} & \text{for } j = 1, ..., k_0, \\ t_{i,j} = t_{k_0} + \dfrac{q_{i,k_0}}{D_i} + (j-1)\dfrac{M_i}{k_i D_i} & \text{for } j = 1, ..., k_i. \end{cases} \end{aligned} \tag{6}$$

If $k_0 > 0$, then each quantity $q_{i,j}$ satisfies equation (2) where q_{i,k_0} depends on M_i.

With respect to Assumptions 6–7, the following condition should be satisfied:

$$\frac{q_{i,k_0} + M_i}{D_i} = T + \frac{q_{i,0}}{D_i} - t_{k_0} = T + \frac{q_0}{D} - t_{k_0} = \frac{q_{k_0} + M}{D},$$

which determines the formula for q_{i,k_0} in (4).

Additionally, there is only one possibility for the initial inventory position, $q_{i,0} = \frac{D_i}{D} q_0$, because q_0 is determined by π. We have $Q - q_0 \sum_{j=1}^{k_0} \lambda^j = M$, as well as formula (5) for $q_{i,0}(\pi)$. Therefore, we have proven the second thesis of the Proposition.

Remark 3. *In the case* $k_0 \geq 1$, *a multi-strategy* $\pi \in \tilde{\Pi}$ *is feasible if the schedule* $q(\pi) = [q_{i,j}]$ *given in Proposition* 1 *satisfies Assumption* 6, *i.e. the condition* $I_i(t) \geq 0$ *for every* $t \in [0, T]$. *In particular, this implies*

$$q_{i,0} \leq \frac{M_i}{k_i} \quad \text{and} \quad \frac{P}{D_i}(\lambda^{k_0} q_{i,0} + \Delta_i) \geq \frac{M_i}{k_i} \quad \text{for every} \quad i = 1, ..., n. \tag{7}$$

The appropriate formulas for the feasibility of π are too complicated to be presented here – see [4] *for a formal definition of* $\tilde{\Pi}$. The set of "strong" feasible multi-strategies will be denoted by

$$\Pi = \left\{ \pi \left| q_{i,0} \leq \frac{M_i}{k_i} \leq \lambda q_{i,k_0} \quad \textit{for} \;\; q(\pi) = [q_{i,j}], k_i > 0, \; i = 1, ..., n \right. \right\} \subset \tilde{\Pi}.$$

Proposition 2. *Let us fix an arbitrary* $\pi = ((Q, k_0), (M_1, k_1), ..., (M_n, k_n))$.
If there exists $q_{i,0} > 0$, $i = 1, ..., n$, (*if* $k_0 \geq 1$ *this is given by Eq.* (5)) *such that*

$$q_{i,0} \leq \frac{M_i}{k_i} \leq \lambda(\lambda^{k_0} q_{i,0} + \Delta_i) = \lambda q_{i,k_0}, \quad \text{for every} \quad i = 1, ..., n, \tag{8}$$

then π is feasible.

Proof. Consider the possibility that the vendor specifies n independent stores in the warehouse (the i-th store for the i-th buyer's demand) produced at individual production rates $P_i = \frac{D_i}{D} P$, $i = 1, ..., n$. For the schedule given in Eq. (4), we have $I_0(t) = \sum_{i=1}^{n} I_0^i(t)$ for each $t \in [0, T]$. It is easy to check that $I_0^i(t) \geq 0$ for each $i = 1, ..., n$ and $t \in [0, T]$ because for some points $t = t_{i,1}$, from (8) we have

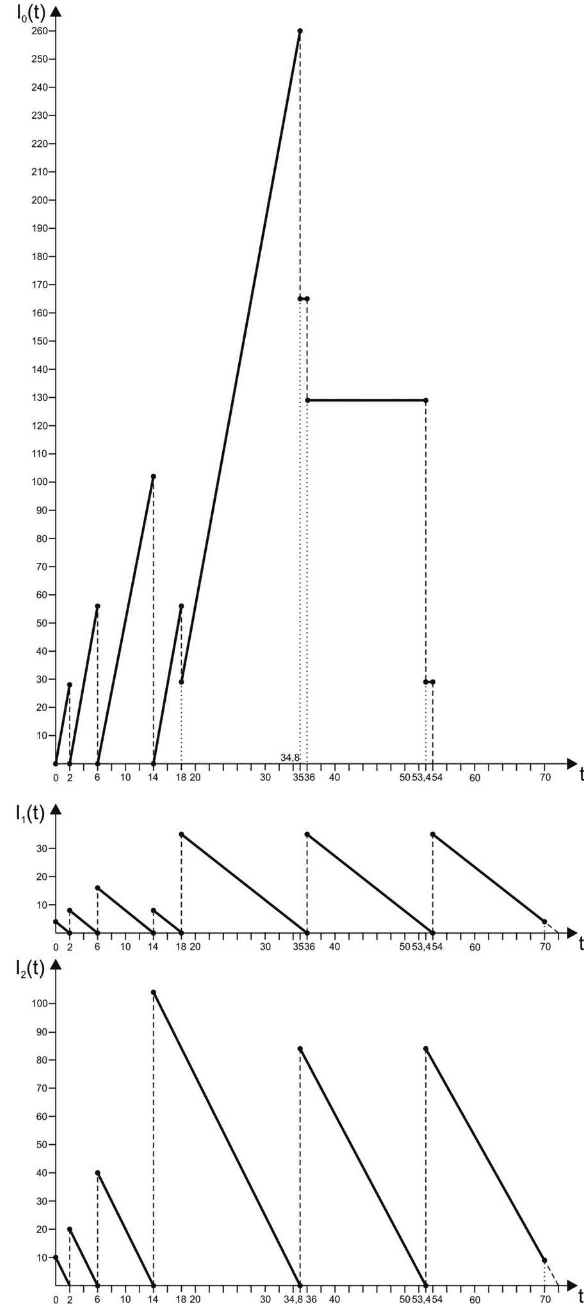

Fig. 2. Inventory positions for the vendor's and buyers' stocks in the PDC given in Example 2 ($n = 2$, $k_0 = 2$, $k_1 = 3$)

$$I_0^i(t_{i,1}) = P_i \frac{q_{i,k_0}}{D_i} = \lambda q_{i,k_0} \geq \frac{M_i}{k_i} \quad \text{for each} \quad i = 1, ..., n. \qquad \square$$

Example 2. *Let us consider the model with two buyers, such that* $(P, D_1, D_2) = (14, 2, 5)$ *and* $\pi = ((Q, k_0), (M_1, k_1), (M_2, k_2)) = ((490, 3), (108, 3), (186, 2))$. *It is easy to check that* $\pi \in \widetilde{\Pi}$ *because of the schedule* $q(\pi)$ *given in Fig.* 2 *and the following Table*: *The stronger feasibility condition* (8) *does not hold*, *i.e.* $\pi \notin \Pi$.

t_j:	t_0 0	t_1 2	t_2 6	t_3 14	$t_{1,1}$ 18	$t_{2,1}$ 34,8	t^* 35	$t_{1,2}$ 36	$t_{2,2}$ 53,4	$t_{2,2}$ 54	T 70
$q_{1,j}$	4	8	16	8	36	–	–	36	–	36	4
$q_{2,j}$	10	20	40	104	–	93	–	–	93	–	10
$I_0(t)$	0	28	56	112	56	255,2	165	165	129	36	0
$I_0(t^+)$	0	0	0	0	20	162,2	165	129	36	0	0

2.3. Cumulative inventories of the agents

Consider a multi-strategy $\pi \in \widetilde{\Pi}$. If $k_0 > 0$, then we have a unique schedule $q(\pi) = [q_{i,j}]$. If $k_0 = 0$, then let us assume that $q(\pi)$ denotes the schedule (4) for given feasible initial inventories $[q_{i,0}]$. Let us denote the cumulative inventories of the agents by $x_i(\pi)$, $i = 0, 1, ..., n$. We have the following formulas: for the vendor

$$x_0(\pi) = \int_0^T I_0(t)dt = x_0'(\pi) + x_0''(\pi), \tag{9}$$

where

$$x_0'(\pi) = \sum_{j=1}^{k_0} \frac{1}{2}(t_j - t_{j-1})q_{i,j} = \begin{cases} 0 & \text{if } k_0 = 0, \\ \dfrac{\lambda(\lambda^{2k_0} - 1)}{2D(\lambda^2 - 1)} q_0^2 & \text{if } k_0 > 0 \end{cases}$$

and

$$x_0''(\pi) = \frac{P}{2}(t^* - t_{k_0})^2 + (t^* - t_{k_0})(T - t^*)P - \sum_{i=1}^{n} \frac{M_i}{k_i} \sum_{j=1}^{k_i} (T - t_{i,j}).$$

For the *i*-th buyer we have:

$$x_i(\pi) = \int_0^T I_i(t)dt = \begin{cases} \frac{1}{2}\frac{M_i^2}{k_i D_i} & \text{if } k_0 = 0, \\ \sum_{j=1}^{k_0-1} \frac{1}{2}(t_{j+1} - t_j) q_{i,j} + \frac{q_{i,k_0}^2}{2D_i} + \frac{M_i^2}{2k_i D_i} & \text{if } k_0 > 0. \end{cases} \tag{10}$$

For the second part of the vendor's cumulative inventory, first we observe that $t^* = t_{k_0} + \frac{M}{P}$ and we use (6) to obtain

$$\sum_{j=1}^{k_i} t_{i,j} = \begin{cases} \frac{k_i q_{i,0}}{D_i} + \frac{k_i - 1}{2D_i} M_i & \text{if } k_0 = 0, \\ k_i \left[\frac{Q - M}{P} + \frac{\lambda^{k_0} q_0 + M}{D} \right] - \frac{k_i + 1}{2D_i} M_i & \text{if } k_0 > 0, \end{cases} \tag{11}$$

for $i = 1, ..., n$. For $k_0 = 0$ we have $t_{k_0} = 0$. Thus we conclude that

$$x_0''(\pi) = \begin{cases} \frac{M}{D}\sum_{i=1}^{n} q_{i,0} + \frac{M^2}{2}\left(\frac{1}{D} - \frac{1}{P}\right) - \frac{M}{2D}\sum_{i=1}^{n}\frac{M_i}{k_i} & \text{if } k_0 = 0, \\ \frac{\lambda^{k_0} q_0 + M}{D} M - \frac{M^2}{2P} - \sum_{i=1}^{n} \frac{k_i + 1}{2k_i D_i} M_i^2 & \text{if } k_0 > 0. \end{cases} \tag{12}$$

We now turn to the buyer's cumulative inventory. From (10), for $k_0 > 0$ we have

$$x_i(\pi) = \frac{1}{2D_i}\left[\frac{\lambda^2(\lambda^{2(k_0-1)} - 1}{\lambda^2 - 1} q_{i,0}^2 + (\lambda^{k_0} q_{i,0} + \Delta_i)^2 + \frac{M_i^2}{k_i} \right]. \tag{13}$$

3. The Stackelberg production-distribution game

3.1. Notation and definitions

We can use the set of strategic characteristics of production-distribution cycles to construct the following constrained $n + 1$ agent game Γ: The vendor uses the strategy (Q, k_0), where Q is total production which is shipped in k_0 deliveries. The i-th buyer uses strategy (M_i, k_i), where M_i is the size of a batch that will be shipped by himself (and he will be charged for that) in k_i identical deliveries. Additionally, the choices of all the agents result in a PDC, i.e. $\pi = ((Q, k_0), (M_1, k_1), (M_n, k_n)) \in \Pi$. Each agent, both the vendor and the buyers, considers the multi-strategy π as a pair $\pi = (\pi_{-i}, \pi_i)$, i.e. as the strategies of the other agents $\pi_{-i} = (...,\pi_{i-1}, \pi_{i+1}, ...)$, together with his own strategy π_i (which he can change according to what the others choose).

The game $\Gamma = (\Pi, \varphi_0, \varphi, V_0, V)$ is established as in [5] by the following definitions of decision rules and cost functions. We have

$$\varphi_0(\pi) = \{(Q, k_0) \mid (\pi_{-0}, (Q, k_0)) \in \Pi\},$$

$$\varphi = (\varphi_1, ..., \varphi_n),$$

where

$$\varphi_i(\pi) = \{(M_i, k_i) \mid (\pi_{-i}, (M_i, k_1)) \in \Pi\}.$$

The cost functions $V = (V_1, ..., V_n)$ are given by the costs (3) and formulas (9)–(13), i.e.:

$$V_i(\pi) = \begin{cases} \dfrac{D}{Q}[A + k_0 A_0 + h_0(x_0'(\pi) + x_0''(\pi)] & \text{for the vendor,} \\ \dfrac{D}{Q}[k_i A_i + h_i x_i(\pi)] & \text{for buyer } i. \end{cases} \tag{14}$$

We will consider the Stackelberg version of the constrained game Γ, denoted by $S\Gamma$, with the vendor as the leader. The vendor chooses his own strategy $\overline{\pi}_0 = (\overline{Q}, \overline{k}_0)$ first. The buyers, knowing the vendor's choice, look for an equilibrium as players in a non-cooperative n person (buyer) constrained game $\Gamma_{\overline{\pi}_0} = (\overline{\pi}_0 \mid \Pi, \varphi, V)$, where

$$\overline{\pi}_0 \mid \Pi = \{(\overline{\pi}_0, \pi_1, ..., \pi_n) \mid (\overline{\pi}_0, \pi_1, ..., \pi_n) \in \Pi_n\} \subset \Pi.$$

Also, for given $\overline{\pi}_0 \in R_+ \times N$ and $\pi \in \Pi$ we will use the notation:

$$\overline{\pi}_0 \mid \pi = (\overline{\pi}_0, \pi_1, ..., \pi_n) \quad \text{and} \quad \overline{\pi}_0 \mid \pi_{-i} = (\overline{\pi}_0, ..., \pi_{i-1}, \pi_{i+1}, ...).$$

Definition 1. *A multi-strategy* $\pi^* \in \overline{\pi}_0 | \Pi$ *is a Nash equilibrium multi-strategy in the constrained game* $\Gamma_{\overline{\pi}_0} = (\overline{\pi}_0 \mid \Pi, \varphi, v)$ *if*

$$v_i(\pi^*) \le v_i(\overline{\pi}_0 \mid \pi^*_{-i}, \pi_i) \text{ for every } \pi_i \in \varphi_i(\pi^*) \text{ and } i = 1, ..., n. \tag{15}$$

We will use the notation $\varepsilon(\overline{\pi}_0)$ *to denote the set of all such equilibrium multi-strategies.*

Definition 2. *A multi-strategy* $\pi^* = (\pi_0^*, \pi_1^*, ..., \pi_n^*) \in \Pi$ *is an equilibrium in the Stackelberg game* $S\Gamma$ *if it is an equilibrium multi-strategy in the game* $\Gamma_{\pi_0^*}$ (i.e. $\pi^* \in \varepsilon(\pi_0^*)$) *and*

$$V_0(\pi^*) \le V_0(\pi) \text{ for every } \pi \text{ such that } (\pi_0 \mid \pi) \in \varepsilon(\pi_0). \tag{16}$$

3.2. Only buyers control all shipments

We consider the case where $k_0 = 0$, i.e. the constrained game $\Gamma^1 = (\Pi^1, \varphi_0, \varphi, V_0, V)$, a sub-game of the game Γ. From (1), we have the following set of multi-strategies:

$$\Pi^1 = \left\{ ((Q, k_0), (M_1, k_1), ..., (M_n, k_n)) \in \Pi \;\middle|\; k_0 = 0, M_i = \frac{D_i}{D} Q, i = 1, ..., n \right\}$$

The constraints are very strong. Namely, for

$$\pi = \left((Q, 0), \left(\frac{D_1}{D} Q, k_1 \right), ..., \left(\frac{D_n}{D} Q, k_n \right) \right) \in \Pi^0$$

the decision rules have the following form: $\varphi_0(\pi) = \{(Q, 0)\}$ and for $i = 1, ..., n$

$$\varphi_i(\pi) = \left\{ \left(\frac{D_i}{D} Q, r \right) \;\middle|\; r \in N \right\}.$$

For the cost functions considered, from (3) and (10) for $k_0 = 0$,

$$V_i(\pi) = \frac{D}{Q} \left[k_i A_i + \frac{h_i}{2} \frac{Q^2 D_i}{k_i D^2} \right] \tag{17}$$

independently of the initial inventory positions $[q_{i,0}]$. The vendor's costs depend on $[q_{i,0}]$:

$$V_0(\pi) = \frac{D}{Q} \left(A + h_0 \left[\frac{Q}{D} \sum_{i=1}^{n} q_{i,0} + \frac{Q^2 (P - D)}{2DP} - \frac{Q}{2D} \sum_{i=1}^{n} \frac{QD_i}{Dk_i} \right] \right) \tag{18}$$

In this sub-game, see Proposition 1 and Remark 3, the schedule $q(\pi) = [q_{i,j}]$ is not unique. From equation (8), the set of possibilities for the initial inventory positions $[q_{i,0}]$ is non-empty. According to Proposition 1, for every feasible collection of $[q_{i,0}]$ we have a PDC schedule. All such PDC schedules have the same strategic characteristic π and the same buyers' costs, given above.

Theorem 1. *There exists a* $Q^* \in R_+$ *and a collection of integers* $\tilde{k}^* = (0,\ k_1^*,\ ...,\ k_n^*)$ *such that the multi-strategy* $\pi^* = ((Q^*, 0), \left(\frac{D_1}{D}Q^*, k_1^*\right), ..., \left(\frac{D_n}{D}Q^*, k_n^*\right))$ *is an equilibrium multi-strategy in the Stackelberg game* $S\Gamma^1$.

Proof. Let us take a quantity $Q \in R_+$. If $\pi \in (Q, 0)|\Pi^1$, then for each $i = 1, ..., n$ the cost function $v_i(\pi)$ depends only on k_i. As a real convex function, it attains its minimum at m_iQ and so we have

$$k_i^*(Q) = \lfloor m_i Q \rfloor, \text{ where } m_i = \sqrt{\frac{h_i D_i}{2A_i D^2}}, \tag{19}$$

and if m_iQ is not an integer, then $\lfloor m_i Q \rfloor$ is defined to be the integer from the two nearest integers to m_iQ which leads to lower costs for the i-th buyer.

Therefore, $\bar{\pi}(Q) = ((Q, 0), \left(\frac{D_i}{D}Q, k_1^*(Q)\right), ..., \left(\frac{D_i}{D}Q, k_n^*(Q)\right))$ satisfies condition (15), formulated in Definition 1 and it is an equilibrium of the game $\Gamma^1_{(Q,0)} = ((Q, 0) \mid \Pi^1, \varphi, V)$ i.e. $\bar{\pi}(Q) \in \mathcal{E}((Q, 0))$.

From (18) and (19), the vendor's costs associated with this equilibrium multi-strategy has the following form

$$V_0(\bar{\pi}(Q)) = \frac{D}{Q}A + h_0\left[\sum_{i=1}^{n} q_{i,0} + \frac{Q(P-D)}{2P} - \frac{1}{2D}\sum_{i=1}^{n}\frac{QD_i}{\lfloor m_i Q \rfloor}\right].$$

It is easy to check (given that the last term is nearly constant), that

$$\frac{\partial V_0}{\partial Q} \approx -\frac{D}{2Q^2}A + h_0\frac{P-D}{2P}$$

and this cost attains a minimum for Q^* when

$$Q^* \approx \sqrt{\frac{PDA}{h_0(P-D)}}. \tag{20}$$

The multi-strategy $\pi^* = \left((Q^*, 0), \left(\frac{D_1}{D}Q^*, k_1(Q^*)\right), ..., \left(\frac{D_n}{D}Q^*, k_n(Q^*)\right)\right)$ satisfies condition (16) for an equilibrium multi-strategy in the Stackelberg game $S\Gamma^1$.

4. The production-distribution sub-game with given numbers of shipments

Let us assume that a sequence of integers $\widetilde{k} = (k_0, k_1, k_n)$, with $k_i > 0$ for each $i = 0, 1, ..., n$, is given (e.g. from negotiations before the game). Additionally, in this section we assume the strong feasibility condition given by (8) in Remark 3.

We define the following restricted game $\Gamma^{\widetilde{k}}(Q)$: The vendor uses strategy $Q > 0$ – total production (chosen first) and the i-th buyer uses strategy $M_i > 0$, $M = \sum_{i=1}^{n} M_i < Q$. $Q - M$ units will be shipped in k_0 initial batches of increasing size. Afterwards M_i units will be shipped in k_i identical batches to buyer i. It is additionally assumed that the multi-strategy is strongly feasible, i.e.

for $s = (s_0, s_1, ..., s_n) = (Q, M_1, ..., M_n)$, we have:

$$S \in Q \mid \Pi^{\widetilde{k}} \Leftrightarrow \pi(s, \widetilde{k}) = ((Q, k_0), (M_1, k_1), ..., (M_N, k_N)) \in \Pi .$$

The constrained game $\Gamma_Q^{\widetilde{k}} = (Q|\Pi^{\widetilde{k}}, \varphi^{\widetilde{k}}, V^{\widetilde{k}})$ is obtained by the projection of

$$\Gamma_{\pi_0} = (\pi_0 \mid \Pi, \varphi, V) \quad \text{from} \quad (R_+ \times N)^{n+1} \quad \text{into} \quad R_+^{n+1} .$$

We have for $i = 1, ..., n$

$$\varphi_i^{\widetilde{k}}(s) = (r \in R_+ \mid (r, s_{-i}) \in \Pi^{\widetilde{k}}\} = \{r \in R_+ \mid (r, k_i) \in \varphi_i(\pi(s, \widetilde{k})\}$$

and, from (13) and (14),

$$V_i^{\widetilde{k}}(M_i, Q \mid s_{-i}) = V_i(\pi(s, \widetilde{k}))$$

$$= \frac{D}{Q}\left\{k_i A_i + \frac{h_i}{2D_i}\left[\frac{\lambda^2(\lambda^{2(k_0-1)} - 1}{\lambda^2 - 1} q_{i,0}^2 + (\lambda^{k_0} q_{i,0} + \Delta_i)^2 + \frac{M_i^2}{k_i}\right]\right\} \tag{21}$$

$$= \frac{D}{Q} k_i A_i + \frac{Dh_i}{2QD_i}\left[\frac{\lambda^2(\lambda^{2k_0-1} - 1}{\lambda^2 - 1} q_{i,0}^2 + 2\lambda^{k_0} q_{i,0}\Delta_i + \Delta_i^2 + \frac{M_i^2}{k_i}\right].$$

From Definition 2, we have

Definition 3. *A multi-strategy* $s^* = (Q^*, M_1^*, ..., M_n^*) \in \Pi^{\widetilde{k}}$ *is an equilibrium multi-strategy in the Stackelberg game* $S\Gamma^{\widetilde{k}}$ *if it is an equilibrium multi-strategy in the game* $\Gamma_Q^{\widetilde{k}}$ *and*

$$V_0^{\tilde{k}}(s^*) \le V_0^{\tilde{k}}(\overline{Q}) \,|\, s) \quad \text{for every } \overline{Q} \quad \text{and} \quad (\overline{Q} \,|\, s) \in \varepsilon(\overline{Q}).$$

Theorem 2. *Let* $\tilde{k}^* = (k_0^*, k_1^*, ..., k_n^*)$ *be a sequence of positive integers. There exists an equilibrium multi-strategy in the production-distribution Stackelberg game* $S\Gamma^{\tilde{k}}$.

Proof. Claim 1. For every $Q > 0$ there exists a Nash equilibrium in the constrained game $\Gamma_Q^{\tilde{k}} = (Q|\Pi^{\tilde{k}}, \varphi^{\tilde{k}}, V^{\tilde{k}})$.

From Proposition 2 in [4], the set of all multi-strategies $\Pi^{\tilde{k}} \cup \{(0, ..., 0)\}$ is a polyhedral convex closed cone with nonempty interior in R_+^{n+1}. Therefore, for every $Q > 0$ the set of multi-strategies $Q \,|\, \Pi^{\tilde{k}}$ is a polyhedral convex closed set in R_+^n. Additionally, the set value functions are upper semi-continuous (for every $s \in Q \,|\, \Pi^{\tilde{k}}$ the set $\varphi_i^{\tilde{k}}(s)$ is a closed interval in R_+).

In the next step, we verify the convexity of the buyers' cost functions. For each buyer $i = 1, ..., n$, from (21) we can transform the cost functions into the form:

$$V_i^{\tilde{k}}(M_i, Q \,|\, s_{-i}) = \alpha_0 + \alpha_1 q_{i,0}^2 + \alpha_2 q_{i,0}\Delta_i + \alpha_3 \Delta_i^2 + \alpha_4 M_i^2, \tag{22}$$

where the α_j are positive and do not depend on the decision variables in the game $\Gamma_Q^{\tilde{k}}$ and from (4)–(5),

$$q_{i,0} = \frac{D_i}{D}\frac{Q - (M_1 + ... + M_n)}{a_{k_0}} \quad \text{and} \quad \Delta_i = \frac{D_i}{D}(M_1 + ... + M_n) - M_i.$$

We compute

$$\frac{\partial V_i^{\tilde{k}}}{\partial M_i}(M_i, Q \,|\, s_{-i})$$
$$= 2\alpha_1 q_{i,0}\left(-\frac{D_i}{Da_{k_0}}\right) + \alpha_2\left[\left(\frac{D_i}{D} - 1\right)q_{i,0} - \Delta_i \frac{D_i}{Da_{k_0}}\right] + 2\alpha_3\left(\frac{D_i}{D} - 1\right)\Delta_i + 2\alpha_4 M_i. \tag{23}$$

It is easy to check that $\frac{\partial^2 V_i^{\tilde{k}}}{\partial M_i^2} > 0.$ This implies the convexity of the cost function $V_i^{\tilde{k}}$ with respect to $M_i \in R_+$. Therefore, from the Arrow–Debreu–Nash Theorem (see [1] p. 182), an equilibrium multi-strategy $s^*(Q) = (Q, M_1^*, ..., M_n^*)$ must exist in the game $\Gamma_Q^{\tilde{k}}$, i.e. $s^*(Q) \in \varepsilon(Q)$.

Claim 2. There exists a best choice for the vendor.

Take a quantity $\overline{Q} > 0$ and, from Claim 1 above, an equilibrium multi-strategy $s^*(\overline{Q}) = (\overline{Q}, M_1^*, ..., M_n^*) \in \varepsilon(\overline{Q})$. Let us denote:

$$\mu^* = (1, \mu_1^*, ..., \mu_n^*), \text{ such that } \mu_i^* = \frac{M_i^*}{\overline{Q}} \text{ for each } i = 1, ..., n.$$

We know that $s^*(\overline{Q}) = \mu^* \cdot \overline{Q}$, as an equilibrium multi-strategy, satisfies the set of inequalities (15), which can be transformed, in the same way as for (22), into the form

$$V_i^{\tilde{k}}(\mu^* \cdot \overline{Q}) = \alpha_0 + \alpha_i^*(\mu_i^* \overline{Q})^2 \le V_i^{\tilde{k}}(\mu_i \overline{Q} \,|\, (\mu^* \cdot \overline{Q})_{-i}) = \alpha_0 + \alpha_i^*(\mu_i \overline{Q})^2 \qquad (24)$$

for each $\mu_i > 0$, such that $\mu_i \overline{Q} \in \varphi_i((\mu^* \cdot \overline{Q}))$ and (8) is satisfied, where the $\alpha_j^* > 0$ do not depend on the decision variables in the games $\Gamma_Q^{\tilde{k}}$.

Analogously, the feasibility conditions given by (8) can be transformed into

$$\beta_i^* \mu_i^* \overline{Q} \le \frac{\mu_i^* \overline{Q}}{k_i} \le \gamma_i^* \mu_i^* \overline{Q}, \qquad (25)$$

where the $\beta_j^*, \gamma_i^* > 0$ do not depend on the decision variables in the games $\Gamma_Q^{\tilde{k}}$.

It is easy to check that for each $Q > 0$ the multi-strategy $\mu^* \cdot Q$ satisfies the set of inequalities (24)–(25). Therefore, for each $Q > 0$ the multi-strategy $\mu^* \cdot Q$ is a Nash equilibrium multi-strategy in the Stackelberg game $\Gamma_Q^{\tilde{k}} = (Q|\ \Pi^{\tilde{k}}, \varphi^{\tilde{k}}, V^{\tilde{k}})$.

The vendor's costs are given by the function $\vartheta(Q) = V_0^{\tilde{k}}(\mu^* \cdot Q)$ and can be transformed, in the same way as above (in (22) and (24)), into

$$\vartheta(Q) = \frac{D}{Q}[A + k_0 A_0 + h_0 \xi Q^2],$$

where $\xi > 0$ does not depend on the decision variables in the games $\Gamma_Q^{\tilde{k}}$.

Therefore, $Q^* = \sqrt{\dfrac{A + k_0 A_0}{2 h_0 \xi}}$ minimizes the cost function $\vartheta(Q)$. From Definition 3, $\mu^* \cdot Q$ is an equilibrium multi-strategy in the game $S\Gamma^{\tilde{k}}$. □

Let us remark that in [5] we can find explicit formulas for the equilibrium strategy of the Stackelberg game $S\Gamma^{\tilde{k}}$ in the one-buyer case.

Theorem 3. *Let* $\widetilde{k}^* = (k_0^*, k_1^*, ..., k_n^*)$ *be a sequence of positive integers. For each* $Q > 0$ *a Nash equilibrium* $s^*(Q) = (Q,\ M_1^*, ..., M_n^*) \in \varepsilon(Q)$ *of the constrained game* $\Gamma_Q^{\widetilde{k}} = (Q|\ \Pi^{\widetilde{k}}, \varphi^{\widetilde{k}}, V^{\widetilde{k}})$ *can be found as a feasible solution of the set of linear equations:*

$$p_i M_i - q_i(M_1 + ... + M_n) = r_i Q \quad \textit{for each} \quad i = 1, ..., n. \tag{26}$$

where the coefficients

$$p_i = Da_{k_0}\frac{k_i + 1}{k_i} - a_{k_0 - 1}D_i, \quad q_i = \frac{D_i}{D}a_{k_0 - 1}\left[D - \frac{1}{\lambda + 1}D_i\right] \quad \textit{and}$$

$$r_i = \frac{D_i}{D}\left[\lambda^{k_0}D - a_{k_0 - 1}\frac{\lambda - 1}{\lambda + 1}D_i\right]$$

do not depend on the decision variables in the game.

Proof. Claim 1. For any $Q > 0$, to find an equilibrium multi-strategy of the game $\Gamma_Q^{\widetilde{k}}$ it is sufficient to find a feasible solution of the set of equations $\frac{\partial V_i^{\widetilde{k}}}{\partial M_i} = 0$, $i = 1, ..., n$. From (21), we have the set of equations

$$q_{i,0}\left[-2\beta\frac{D_i}{Da_{k_0}} + 2\lambda^{k_0}\left(\frac{D_i}{D} - 1\right)\right] + 2\Delta_i\left[-\frac{\lambda^{k_0}D_i}{Da_{k_0}} + \frac{D_i}{D} - 1\right] + \frac{2M_i}{k_i} = 0, \tag{27}$$

where $\beta = \frac{\lambda^2(\lambda^{2k_0} - 1)}{\lambda^2 - 1}$, $i = 1, ..., n$.

The set of equations (27) is equivalent to

$$\frac{D_i(Q - M)}{Da_{k_0}}[a_{k_0}\lambda^{k_0}(D_i - D) - \beta D_i]$$

$$+\left(\frac{D_i M}{D} - M_i\right)[a_{k_0}(D_i - D) - \lambda^{k_0}D_i] + Da_{k_0}\frac{M_i}{k_i} = 0 \quad \text{for} \quad i = 1, ..., n.$$

Therefore, (27) is equivalent to

$$[a_{k_0}(D - D_i) + \lambda^{k_0}D_i]M_i - \frac{D_i}{D}[D_i(\beta - \lambda^{k_0}) + (D - D_i)(\lambda^{k_0} - a_{k_0})]M$$

$$= \frac{D_i}{D}[D_i(\beta - \lambda^{k_0}) + \lambda^{k_0}D] \quad \text{for} \quad i = 1, ..., n.$$

The theorem follows from this, since

$$\beta - \lambda^{k_0} = -a_{k_0-1}\frac{\lambda-1}{\lambda+1} \quad \text{and} \quad \lambda^{k_0} - a_{k_0} = -a_{k_0-1}. \qquad \square$$

Corollary. *In the case where n* = 2, *for a given Q* > 0 *we obtain a set of linear equations* (26) *of the form:*

$$\begin{cases}(p_1-q_1)M_1+q_1M_2=r_1Q\\ q_2M_1+(p_2-q_2)M_2=r_2Q.\end{cases}$$

The solution of this system is

$$\begin{cases}M_1=\dfrac{(p_2-q_2)r_1+q_1r_2}{(p_1-q_1)(p_2-q_2)-q_1q_2}Q\\[2ex] M_2=\dfrac{(p_1-q_1)r_2+q_2r_1}{(p_1-q_1)(p_2-q_2)-q_1q_2}Q.\end{cases}$$

5. Final remarks

The main goal of this paper was to show that satisfactory coordination of inventories can be achieved using a competitive approach, such as the framework of non-cooperative games. Such games are different under this competitive regime, i.e. in the classes of admissible policies indexed by $\tilde{k}$. In such games, the agents independently choose strategies to minimize their costs. The equilibria multi-strategies in such games, as well as the total costs and agents' participation in such costs, are different. In numerical simulations for the case with one buyer, we observe that the total costs of optimal cooperative policies are close to the total costs of equilibrium strategies under the condition that their regimes are of the same form as optimal policies in the cooperative case. Theoretical justification of this observation in the multi-buyer case is beyond the scope of this paper. One question still unanswered is whether there exists an equilibrium in the main constrained game Γ or the Stackelberg game $S\Gamma$.

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