NON-STATIONARY STOCHASTIC SEQUENCES AS SOLUTIONS TO ILL-POSED PROBLEMS

RICHARD HORSKÝ
University of Economics in Prague, Faculty of Statistics and Informatics, Department of Mathematics, Ekonomická 957, 148 00 Praha 4, Czech Republic
email: rhorsky@vse.cz

Abstract

The mathematical formulation of any problem in applied sciences leads usually to an operator equation. In linear models the operator is a matrix or a difference, differential or an integral operator. If we try to solve such an operator equation we often meet different difficulties. The solution need not exist or it is not unique or it is unstable. All these bad properties of a mathematical model reflect the complexity of the real problem we are to solve. Such difficult problems are called ill-posed problems. It is not surprising that we also meet these problems in time series analysis. In this area the ill-posedness is demonstrated by the non-stationarity of a stochastic process. In economics it is well-known that the time series of logaritms of GDP or the price of overall market portfolio are almost certainly non-stationary. The typical example of the non stationary process is the random walk.

Key words: *ill-posed problem, regularization, non-stationary sequence, differencing, lag operator, least square solution, random walk, GDP, market portfolio*

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1. Introduction

In 1902 Jacques Hadamard formulated what is the well-posed problem when he studied the partial differential equations. He also gave today classical example of the problem which is not well-posed: the Cauchy problem for Laplace`s equation. Hadamard`s concept reflected the idea that any mathematical model of a physical phenomena should have the following properties

1. There exists a solution to the problem (existence).
2. There is only one solution to the problem (uniqueness).
3. The solution depends continuously on the data (stability).

If one of these conditions is violated the problem is called ill-posed. We can give a simple example: the system of linear algebraic equations $Ax = b$ with the square matrix *A*. This problem is well-posed if *A* is a nonsingular. In other case the conditions 1 or 2 above are violated. As to the condition 3 a square matrix represents always a continuous mapping because in the finite dimensional spaces any linear operator is continuous. In spite of this we can watch a certain kind of instability: it is caused by the fact that the matrix *A* is ill-conditioned. The notion of ill-condition matrix is well-known from the numerical mathematics.

However the ill-posed problems began to be studied intensively as late as in 1950`s. Here it is necessary to commemorate above all the outstanding Russian mathematicians A. N. Tikhonov, M. M. Lavrentiev, V. K. Ivanov and their disciples, e.g. V. A. Morozov, who laid the foundations of approximate methods for solving ill-posed problems and contributed much to the development of the theory and techniques which became new, interesting and fruitful area in numerical analysis.

The ill-posed problem has to be regularized to get some reasonable results. There exist different methods how to do that. We can recall for instance the truncated singular value decomposition or well-known and favorite the Tikhonov regularization method which was introduced in early 1960´ by A. N. Tikhonov.

The non-stationary stochastic processes generate non-stationary time series. These series represent a serious problem for their instability. Unfortunately they occur very often in economic and financial practice. Let Y_t be the GDP of some economy at time *t*. Then $y_t = \log Y_t$ is almost certainly non-stationary series. Similarly if S_t is the price of overall market portfolio then $s_t = \log S_t$ is again almost certainly non-stationary. The non-stationary process is unstable which leads to poor results in forecasting. The typical example of the non-stationary process is the random walk. Other non-stationary processes denoted in Box Jenkins methodology as ARIMA processes involve the random walk. The random walk can be defined as a solution of the stochastic difference equation of the first order. This equation is in fact an ill-posed operator equation. We employ the access and methods of the functional analysis to treat it

2. Ill-posed Problems

The classical definition of the ill-posed problem was given above in introduction. Due to the powerful computers the theory of ill-posed problems has become widely used in solving problems in various branches of science including economics (Horowitz, 2014, Hoderlein, Holzmann, 2011, Lu, Mathe, 2014). In mathematics we see many examples of the ill-posed problems in arithmetic (division by a small number), linear algebra (solving the system of linear algebraic equation with a singular matrix), calculus (differentiation), integral equations (Fredholm equation of the first kind) or functional analysis (the invertibility of a compact operator). We can find these problems in time series analysis (Sanchez, 2002). The ill-posedness is in a close relation to the problem of overdifferencing of a time series (Bell, 1987).

As was mentioned the mathematical formulation of a problem in applied sciences is usually of the form of an operator equation

$$Ax = b \,, \tag{1}$$

where $A : V \to W$ is a given operator defined on a given space *V* with values in a given space *W*, *b* is a given element in *W*. The spaces *V* and *W* respectively are equipped with some convergence structures. They are usually linear normed spaces. Then the equation (1) is the ill-posed problem if and only if *A* violates one the following properties:

1. *A* is bijective (it means it is a one-to-one correspondence between both the spaces *V* and *W*).
2. $A^{-1} : W \to V$ is continuous (if it exists of course).

In the following section we perform two typical examples of the ill-posed equation (1).

2.1 Examples of ill-posed problems

In the finite dimension the operator *A* is represented by a matrix **A** Thus the equation (1) is represented by the system of linear algebraic equations $\mathbf{Ax} = \mathbf{b}$. If the matrix **A** is singular the operator *A* is not bijective (in fact it is neither surjective nor injective). It means that the properties 1 and 2 from the Hadamard´s definition of the well-posed problem (given above in section 1) are violated. As to the third property a linear operator in finite dimensional spaces is always bounded (continuous). However a certain kind of instability can occur even if **A** is a nonsingular matrix. This is the case of numerical instability caused by a large condition number of this matrix $\operatorname{cond} \mathbf{A} = \|\mathbf{A}\|\|\mathbf{A}^{-1}\|$. Let us give an example. The lag operator known in time series analysis (eg. Arlt, 1999 , Dhrymes 1980 or Horský, 2013) is represented in finite dimensional case by the matrix

$$\mathbf{B}_N = \begin{bmatrix} 0 & 1 & 0 & & 0 \\ 0 & 0 & 1 & & 0 \\ & & & & \\ 0 & 0 & & 0 & 1 \\ 0 & 0 & & 0 & 0 \end{bmatrix}. \tag{2}$$

The matrix (2) is a singular (even nilpotent) matrix of the order *N*. On the other hand the matrix of the difference operator in the Euclidean space of the dimension *N* is nonsingular

$$\mathbf{I}_N - \mathbf{B}_N = \begin{bmatrix} 1 & -1 & 0 & & 0 \\ 0 & 1 & -1 & & 0 \\ & & & & \\ 0 & 0 & & 1 & -1 \\ 0 & 0 & & 0 & 1 \end{bmatrix}.$$

Both ${}_{\infty}$ norm and ${}_{1}$ norm of this matrix is equal to 2. Since (2) is nilpotent we easily obtain its inverse

$$(\mathbf{I}_N - \mathbf{B}_N)^{-1} = \mathbf{I}_N + \mathbf{B}_N + \mathbf{B}_N^2 + \quad + \mathbf{B}_N^{N-1} = \begin{bmatrix} 1 & 1 & 1 & & 1 \\ 0 & 1 & 1 & & 1 \\ & & & & \\ 0 & 0 & & 1 & 1 \\ 0 & 0 & & 0 & 1 \end{bmatrix}$$

the norms of which are equal to *N*. Hence $\operatorname{cond}(\mathbf{I}_N - \mathbf{B}_N) = 2N$ and the matrix of the difference operator is ill-conditioned for large *N*.

In the infinite dimension the linear operator need not be bounded. In spite of the fact that the compact operator (the operator which maps any compact set to a relative compact set, see Lukeš 2012) is bounded, its inverse (if it exists) may not be bounded because the compact operators form two-sided ideal in the space of all bounded operators on a given space (of infinite dimension) and the identity is not compact operator in infinite dimensional space. Let us give an example again. The Volterra operator

$$y(t) = Ax(t) = \int_0^t x(s)ds, \tag{3}$$

where $t \in [0,1]$, $x \in V = W = C[0,1]$ (the space of all continuous functions with the supremum norm is Banach space) represents the operation of indefinite integration. It is known (see Lukeš, 2012) that this operator is compact. Its range $R(A)$ is the space of all functions in $C[0,1]$ having the continuous derivative in $(0,1)$ satisfying $y(0) = 0$. The inverse operator is defined on $R(A)$ and it is the derivative. The derivative is not bounded operator, it is only so called closed operator and $R(A)$ is an open subspace of $C[0,1]$. It follows the differentiation is ill-posed problem which may be surprising if we compare the symbolic differentiation and symbolic integration.

Another case of ill-posed problem in infinite dimensional spaces is regarded below.

2.2 The General Concept of Regularization

Suppose again the general problem (1), where $A: V \to W$ **,** *V* and *W* are normed linear spaces, and *A* is a bounded linear and injective operator with inverse which is not bounded, i.e. (1) is ill-posed problem. This is a situation which is usual in applications and we met it in the example with the operator (3). The problem (1) is unstable and the idea is to approximate this problem by a well posed one. The procedure for this replacement is called *regularization method*.

First we deal with approximations of the inverse operator A^{-1} (which is not bounded). A *regularization family for A* is a collection of bounded linear operators $R_\alpha : W \to V,\ \alpha > 0$, for which $\lim_{\alpha \to 0+} \|R_\alpha Ax - x\| = 0$ for all $x \in V$. A regularization family may not be uniformly bounded. In fact, there is a sequence of positive numbers $\alpha_n \to 0$ such that $\|R_{\alpha_n}\| \to \infty$. If there were a constant *c* such that for all $\alpha > 0$ **,** we would have $\|R_\alpha\| \le c$ **,** then for any $y \in R(A)$ the following inequalities would hold:

$$\left\|A^{-1}y\right\| \le \left\|A^{-1}y - R_\alpha y\right\| + \|R_\alpha y\| \le \|x - R_\alpha Ax\| + \|R_\alpha\|\|y\| \le \|x - R_\alpha x\| + c\|y\| \tag{4}$$

The first term on the right hand side in (4) tends to zero as $\alpha \to 0$. However, it means that $\left\|A^{-1}y\right\| \le c\|y\|$ which is a contradiction with A^{-1} is not bounded. In other words, the product $R_\alpha A$ converges to identity *I* pointwise, not uniformly.

Second the right hand side of the equation (1) contains a noise. We could watch the effect of this noise in the third example in the previous section. Suppose that $\boldsymbol{b_\delta}$ is a perturbation of *b* in (1), $\|b - b_\delta\|_\infty \le \delta$**.** We then define

$$x_{\alpha,\delta} = R_\alpha b_\delta. \tag{5}$$

The vector (5) is called the *regularized solution to (1).*

Let $x^* = A^{-1}b$ is the exact solution to the problem (1). Now we derive the fundamental estimate for the error of the regularized solution

$$\left\|x^* - x_{\alpha,\delta}\right\| \le \left\|x^* - R_\alpha b\right\| + \|R_\alpha b - R_\alpha b_\delta\| \le \left\|x^* - R_\alpha Ax^*\right\| + \|R_\alpha\|\delta. \tag{6}$$

As α tends to zero the first term in the right hand side in (6) converges to zero (*regularization effect*) while the second term grows to infinity (*ill-posedness effect*). Thus two competing effects enter (6). These effects forces us to make a trade off between accuracy and stability. The basic question is how to choose the value of the parameter α which is dependent on the given $\delta > 0$ (and also on $b_\delta \in W$**).**

2.3 The Generalized Solution (GSLS)

The exact solution to the operator equation is usually called the classical solution. However it need not exist. If $b \notin R(A)$ one can try to look for a solution in a generalized sense. It is natural to require that such a solution minimizes the residual norm $\|Ax - b\|$. The following notions and considerations are performed when both spaces *V* and *W* are Hilbert spaces: The vector $x_0 \in V$ (if it exists) for which

$$\|Ax_0 - b\| = \inf_{x \in V} \|Ax - b\| \tag{7}$$

is called the *least square solution* or the *generalized solution to the equation (1) in the sense of the least squares* (GSLS). However it need not exist. The sufficient condition for the existence of the GSLS is that the range *R(A)* is a closed subspace of *W.* It will be the case if for example the space *R(A)* is of a finite dimension. On the other hand if *R(A)* is a dense subspace of *W* different from *W* there is always an element $b \in R(A)$ such that the equation (1) has no GSLS.

It is the known fact that $x_0 \in V$ is a GSLS of (1) if and only if it is the solution to the *normal form of the equation (1)*

$$A^* Ax = A^* b \,, \tag{8}$$

where A^* is the adjoint operator to the operator *A*. The optimization problem (7) is thus equivalently reformulated as the linear equation (8). If GSLS exists it need not be unique. All the solution of (8) form a linear set $L_b = \{x \in V : A^* Ax = A^* b\} = x_0 + N(A)$, where $N(A)$ is the kernel of the operator *A*. To assure the uniqueness of the solution to (8) we impose some additional condition. It is usually the requirement to the least norm of an element within the set L_b: if $L_b \neq \varnothing$ there exists a unique element $\widetilde{x} \in L_b$ for which $\|\widetilde{x}\| = \inf_{x \in L_b} \|x\|$. We can notice that the new problem (8) is an operator equation with a self-adjoint positive operator. That is why we talk about the symmetrization of the original problem (1).

3. The Stochastic Sequences

The non-stationary time series are generated by non-stationary stochastic processes. These processes can be understood as a solutions of operator equations which are classified as ill-posed problems. We will introduce an application of the Tikhonov regularization method to the essential problem the solution of which is well-known random walk. The other non-stationary models in Box Jenkins methodology involve this case. In time series analysis the classical access to stabilization is the differencing of the time series. It can be also taken as a certain type of regularization.

The stochastic process is defined as a mapping $\mathbf{X} : T \to L_2$, where the set *T* (time domain) is here set of all integers and $L_2 = L_2(\Omega, \pi)$ is the space of all functions (random variables) defined almost everywhere on the measurable space **Ω**, that are square integrable over the space **Ω** with respect to a probability measure π. It means that this mapping is in fact a (both sided) sequence. However we adopt slightly different modification of this current definition which consists in the assumption that the sequence is actually one sided of the type

$$\mathbf{X} = (X_t, X_{t-1}, \quad) = (X_{t-k}), k \in \mathbf{N}_0, t \in T. \tag{9}$$

If we extend the stochastic sequences (9) by zeros to time points *t*+1, *t*+2,... we get the both sided stochastic sequence from the original definition given above.
Basic characteristics of the stochastic sequences are as follows:

- the function of means $\mu_t = EX_t$;
- the function of variances $\sigma_t^2 = DX_t = EX_t^2 - \mu_t^2$;
- the autocovariance function $C(t,s) = \mathrm{cov}(X_t, X_s) = E(X_t - \mu_t)(X_s - \mu_s)$.

The definitions of the *stationary process,* the *white noise process* and the *general linear process,* are given for instance in Arlt (1999). Let us recall that the stationary process is such a process for which the functions of means and variances are constant and the values of its autocovariance function depend only on the time lag of its terms. The sequences (9) are elements of certain spaces. Instead of the Banach space of all bounded stochastic sequences $\ell_\infty(L_2)$ (see Horský, 2016) we will consider now the space $_2(L_2)$, which is the space of all square summable stochastic sequences. The framework of the Hilbert space $_2(L_2)$ will be more suitable for our purposes.

3.1 Stochastic Sequence as an Element of the Hilbert Coordinate Space

All the terms of the stochastic sequence (10) are elements of the Hilbert space $L_2 = L_2(\Omega, \pi)$. Their norms in L_2 are $\|X_{t-k}\|_2^2 = EX_{t-k}^2 = \sigma_{t-k}^2 + \mu_{t-k}^2$. If (10) is the centered (with zero mean) and stationary sequence then its terms lie on the sphere with radius σ^2 in L_2. The space of all bounded stochastic sequences $_\infty(L_2)$ with the norm $\|\mathbf{X}\|_\infty = \sup_{k\geq 0} \sqrt{EX_{t-k}^2}$ was used in Horský (2016). It was natural since this space contains all stationary sequences and it is the Banach space. However in what follows we will employ another space. The space $_2(L_2)$ of all stochastic sequences (9) which satisfy $\sum_{k=0}^{\infty} EX_{t-k}^2 < \infty$ is the Hilbert space with respect to the norm derived from the dot product $\langle \mathbf{X}, \mathbf{Y} \rangle = \sum_{k=0}^{\infty} E(X_{t-k}Y_{t-k})$, i.e. the norm $\|\mathbf{X}\| = \left(\sum_{k=0}^{\infty} EX_{t-k}^2 \right)^{1/2}$.

Unfortunately the space $_2(L_2)$ contains no stationary stochastic sequences as well as the constant sequences (except the zero sequence). In spite of this apparent drawback this space has more comfortable structure for our intentions than the space $_\infty(L_2)$. We may overcome this deficiency if we slightly modify the concepts of stationarity of the process and white noise respectively. As to the mean we will consider only centered sequences. We allow the sequence to be stationary as far as to the past as one wishes then the norm (variance) of its terms has to fall to zero (otherwise the series of variances may not be convergent).

3.2 Lag Operator and Its Spectral Properties

The lag operator is defined for example in Dhrymes (1985). We will take this operator in the Hilbert space $_2(L_2)$ and thus

$$B:\ _2(L_2) \to\ _2(L_2),\ B(X_{t-k}) = (X_{t-1-k}) \tag{10}$$

It is a bounded operator with the norm equal to one, i.e. the same value when this operator is defined on the space $_\infty(L_2)$ (see Horský, 2013).

The spectral properties of the lag operator in the space $_\infty(L_2)$ are summarized in Horský (2016). We can recall that the spectrum of the lag operator is the closed unit circle in the complex plane and all elements of this circle are the eigenvalues of this operator. In the case of (11) it is almost the same thing. There is only one difference: if λ is a complex unit, i.e. $|\lambda| = 1$ then the operator $(\lambda I - B)^{-1}$ exists but it is not bounded. It follows that the operator $(\lambda I - B)^{-1}$ may not be defined on the whole Hilbert space $_2(L_2)$ but only on some proper subspace of this space. In fact $R(\lambda I - B)$ is a proper subspace of $_2(L_2)$ which is dense in $_2(L_2)$. If we take for example $\lambda = 1$ the corresponded eigenvector in $_\infty(L_2)$ is $\mathbf{E} = (1,1,1, \ )$ but obviously $\mathbf{E} \notin\ _2(L_2)$.

There is an important class of stochastic sequences which are the solutions to the stochastic difference equation of the type

$$\Phi(B)\mathbf{X} = \Theta(B)\mathbf{W} , \tag{11}$$

where $\Phi(B)$ and $\Theta(B)$ are polynomials in B of the degree p and q respectively, and $\mathbf{W}$ is a white noise. As was shown in Horský (2016) the operator equation (11) is the well-posed problem if and only if all the roots of the polynomial $\Phi(z)$ are outside the closed unit circle in the complex plane. In this case the solution to (12) is well-known ARMA(*p,q*) process. On the other hand if $\Phi(1)=0$ then $\Phi(B)$ is either not invertible or it has inverse which is not bounded. In any case (11) is then an ill-posed problem. In terminology of Box Jenkins methodology the solution to (11), if it exists, is of the class ARIMA(*p,d,q*), where *d* is the multiplicity of the unit as a root of the $\Phi(z)$. Such a sequence is the *non-stationary stochastic process*.

4. The Random Walk as the Solution to an Ill-posed Problem

If we set in (11) $\Phi(B):=I-B$, i.e. $\Phi(B)$ is the difference operator, and $\Theta(B):=I$ respectively, then we get the essential form of the ill-posed equation (11), i.e.

$$(I-B)\mathbf{X}=\mathbf{W}\ . \tag{12}$$

The equation (12) is the stochastic difference equation of the first order. We will deal with the problem of the regularization of its solution in the Hilbert space $_2(L_2)$ by the Tikhonov regularization method. The solution to the operator equation (12), if it exists, is called the *random walk process.*

4.1 The Least Square Solution to the Equation $(I-B)\mathbf{X}=\mathbf{W}$

The adjoint operator to the difference operator $(I-B)$ is obviously $I-B^*$ and the normal form (8) of the equation (12) is

$$\left(I-B-B^*-B^*B\right)\mathbf{X}=\left(I-B^*\right)\mathbf{W}. \tag{13}$$

For simple writings let us set $A=(I-B)$. It follows from the relations among the kernels and the ranges respectively of mutually inverse operators (see Lukeš 2012) that we have $N\left(A^*A\right)=N(A)=\{o\}, \overline{R\left(A^*A\right)}=\overline{R\left(A^*\right)}= {}_2\left(L_2\right), R\left(A^*A\right)\subseteq R\left(A^*\right)\subset {}_2\left(L_2\right)$. If we add the spectral properties of the adjoint operator B^* we see that the equation (13) is an ill-posed problem as well as (12). Thus the GSLS to (12) need not exist for some right side (from the space $_2(L_2)$).

4.2 Tikhonov regularization on the equation $(I-B)\mathbf{X}=\mathbf{W}$

The Tikhonov regularization consists in the regularization of the normal form of the given problem. We keep the denotation from the previous section, i.e. $A=I-B$. The basic idea of the Tikhonov regularization is to add some positive multiple of the identity αI to the operator A^*A. This way we get a bounded linear bijection $A^*A+\alpha I$ on the space $_2(L_2)$. The regularized form of the equation (13) is

$$\left(I-B-B^*-B^*B+\alpha I\right)\mathbf{X}=\left(I-B^*\right)\mathbf{W}. \tag{14}$$

The equation (14) is well-posed. The collection of operators $R_\alpha=\left(A^*A+\alpha I\right)^{-1}A^*$ is a regularization family for the operator A^*A. As in the section 1.2 we replace in (14) the white noise $\mathbf{W}$ by its perturbation $\mathbf{W_\delta}$, $\left\|\mathbf{W}-\mathbf{W}_\delta\right\|\le\delta,\ \delta>0$. Then we get the regularized solution $\mathbf{X}_{\alpha,\delta}=R_\alpha\mathbf{W}_\delta$.

Let us describe the fundamental estimate for the error of the Tikhonov method. First we estimate the norm $\left\|R_\alpha\right\|$. For this we use the polar decomposition of a bounded linear operator (see Lukeš, 2012) and the Riesz functional calculus (see Lukeš, 2012). By polar decomposition $A = U|A|$, where U is an isometry, in our case even unitary operator, and $|A|$ is the square root of the operator $T := A^* A$ which is self-adjoint. Hence we have $R_\alpha = (T + \alpha I)^{-1}|A|U$ and if we denote $f(T) = (T + \alpha I)^{-1}|A|$ we obtain $\|R_\alpha\| \le \|f(T)\|$. To compute the norm of the self-adjoint operator $f(T)$ we employ the Riesz functional calculus

$$\|f(T)\| = \sup_{\lambda\in[0,4]}|f(\lambda)| = \sup_{\lambda\in[0,4]}\frac{\sqrt{\lambda}}{\lambda+\alpha} = \frac{1}{2\sqrt{\alpha}}. \tag{15}$$

The upper bound 4 in the interval in (15) is obtained from the fact that $\|A\| = \|I - B\| = 2$ (the spectrum of the difference operator is only translated spectrum of the lag operator, which follows from the spectral mapping theorem, and $\|T\| = \|A\|^2$ (see Lukeš 2012).

Second we estimate the regularization effect $\|\tilde{\mathbf{X}} - R_\alpha A\tilde{\mathbf{X}}\|$. Here we will suppose that $\tilde{\mathbf{X}} = A^*\mathbf{Z}$ for some $\mathbf{Z} \in {}_2(L_2)$, i.e. the GSLS (the exact solution to (13)) belongs to the range of the adjoint operator A^*. This can be interpreted as a certain requirment on smoothness of the GSLS. For comfortable writing set $G_\alpha := (T + \alpha I)^{-1}$. We derive

$$\begin{aligned}\|R_\alpha A\tilde{\mathbf{X}} - \tilde{\mathbf{X}}\| &= \|G_\alpha T\tilde{\mathbf{X}} - \tilde{\mathbf{X}}\| = \|G_\alpha(T\tilde{\mathbf{X}} - \alpha\tilde{\mathbf{X}} - T\tilde{\mathbf{X}})\| = \alpha\|G_\alpha\tilde{\mathbf{X}}\| = \alpha\|G_\alpha A^*\mathbf{Z}\| \le \\ &\le \alpha\|R_\alpha\|\|\mathbf{Z}\| \le \frac{\alpha}{2\sqrt{\alpha}}\|\mathbf{Z}\| = \frac{\sqrt{\alpha}}{2}\|\mathbf{Z}\|.\end{aligned} \tag{16}$$

Finally we join the two estimations (15) and (16) and thus get the fundamental estimation for the error of the Tikhonov method:

$$\|\tilde{\mathbf{X}} - \mathbf{X}_{\alpha,\delta}\| \le \|(A^*)^{-1}\tilde{\mathbf{X}}\|\frac{\sqrt{\alpha}}{2} + \frac{\delta}{2\sqrt{\alpha}}. \tag{17}$$

The first term on the right hand side in (17) reflects the regularization effect while the second one the ill-posedness effect. We have to balance the parameters α and δ in such a manner so that $\frac{\delta}{\sqrt{\alpha}} \to 0$ if $\alpha \to 0_+$ and, at the same time, $\delta \to 0_+$. The rule for the choice of the value of the parameter α may be so called Morozov discrepancy principle (eg. Morozov, 1984 or Nair 2009). It consists in taking $\alpha = \alpha(\delta)$ such that $\|A\mathbf{X}_{\alpha,\delta} - b_\delta\| = \delta$. This choice guarantees the balanced approximation and data noise errors. The numerical computation of α can be carried out by Newton´s method.

The biggest drawback of Tikhonov regularization is the repeated matrix manipulation, computation of the inverses $G_\alpha := (T + \alpha I)^{-1}$.

5. Conclusion

The polynomial operators in the equation (11) may have a unit root. It causes either non-stationarity or non-invertibility of this model. In essential form the unit root is contained in the model (12). In such case the equation (11) is an ill-posed problem. Then it should be solved by a regularization method if we wished to obtain some reasonable solution to this problem.

As the spectral analysis of the lag operator in ${}_2(L_2)$ shows it remains to prove the existence of the exact solution (in some sense) to the original problem (12) since the range of the difference is not closed and not equal to ${}_2(L_2)$; it is only dense subspace of ${}_2(L_2)$.

The Tikhonov regularization method raise again the question about the overdifferencing of time series since the transformation of the equation (12) to its normal form contains in fact the second difference of the process.

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