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D I D A C T I C S O F M A T H E M A T I C S

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EFFECTIVENESS OF MATHEMATICAL EDUCATION

Abstract. In the paper, the measure of teaching quality is introduced. This measure depends on distribution and average level of students' knowledge. It has been applied and verified on the basis of the data concerning first-year students of the Wrocław University of Economics (results of secondary school certificate in mathematics, exam in mathematics at the University and the survey conducted at the end of the course in mathematics have been taken into account). Results for Poland, concerning effectiveness and efficiency of teaching, have been compared to other countries.

Key words: measurement, effectiveness and efficiency of teaching.

1. Introduction

Mathematics is not only a way of thinking, but also a fundamental descriptor of nearly every other field of knowledge studied in modern science. The role of mathematics and the proper position of the field in the hierarchy of education subjects have been emphasized not only by scholars of science and technology, but also in the field of humanities, such as philosophy and theology. As put by the Reverend professor A. Szostek, *proper humanistic background requires good level of mathematical education*.

The term mathematics means knowledge. In ancient times, there was no knowledge outside mathematics; it is a study of physical world in its most abstract form. A theory devoid of material equivalents is nothing more than a futile deliberation. Universalism of mathematics is a product of abstraction. Mathematics is predominantly based on deduction, but, since deduction must be based on verifiable facts, mathematics is also a field of empirical studies (A. Smoluk (2007)).

To improve the effectiveness of a process, such as mathematical education, it is necessary to start with a proper diagnosis (the need for realism), and follow up with a suitable and well-maintained remedial programme (as in disease treatment). It seems that present attempts to improve the effectiveness of mathematical education lack one fundamental element: that of realistic expectations. The intellectual potential of students may be harmed not only by low expectations, but also by unrealistically high level of requirements (contrasted with student skill level).

2. Realism

In his book *Réflexions sur la conduite de la vie*, A. Carrel (the youngest laureate of the Nobel Prize in medicine) put the following remark: "Little observation and lots of reasoning leads to errors. Lots of observation and little reasoning leads to truth". St. Augustine expressed a similar view by saying: "I seek to learn and know, and not to think about it". He also warns that the opposite to that thesis is not valid, since we tend to ascribe our thoughts to facts, even if involuntarily.

The effectiveness of education depends on the quality of teaching process, described by such elements as curriculum (adapted to the intellectual level of the student and to the requirements of labour market), teaching personnel and equipment. On the other hand, however, it is also related to knowledge, skills and behaviour of students.

The construction of mathematical syllabus should be based on well-documented postulates (T. Szapiro (2007)):

1. Education should aim to develop the following skills: thinking, reasoning, gathering knowledge and the ability to apply the knowledge in practice.

2. Education should aim to incorporate all the above elements in a flexible manner, adjusting their proportions according to the need and level of education process.

3. Education should be effective – it should define goals and meet them.

4. Education should be efficient – it should not waste resources and prospects.

J. Łyko documented this lack of realism in his study of design and realization of mathematical calculus in the curriculum of Wrocław University of Economics. Among other data, the author emphasizes drastic reduction of teaching hours: from 120 per year down to 45 in the subsequent academic year, within the same scope of material and with notably decreasing skill of students (!).

This academic year, 66% of fresh students enrolled in the department of Economic Sciences do not even have a basic school-leaving examination in mathematics. This lack of fundamental evaluation of mathematical skills results not only in skill deficiency, but also in fear of independent thinking and even fear of contact with mathematical handbooks.

Based on a survey of 1st year students of this department (Economic Sciences) conducted at the end of 2nd semester (2008), as well as results of examinations, this paper will present the problem at hand and suggest curative measures.

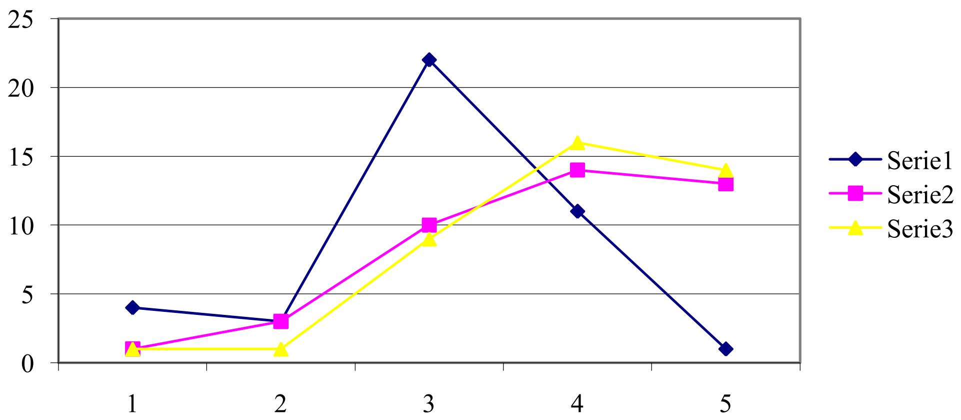

Fig. 1. Survey responses

Source: own research.

1. Does the instructor present material in a clear way?
2. Does the instructor show enthusiasm in teaching?
3. Does the instructor promote active participation in lectures?

Respondents were asked to evaluate their lectures based on 6 questions, on a scale from 1 to 5. “Correct” (ideal) evaluation charts should show left-side asymmetry (good lecture presentation) and small variability coefficient (positive evaluation on the part of majority of respondents). Two of the six queries (“Does the instructor present material in a clear way?” and ”Utility of references suggested at lecture”) contrast with the above rule; those two queries are ranked low and display the greatest variability coefficient.

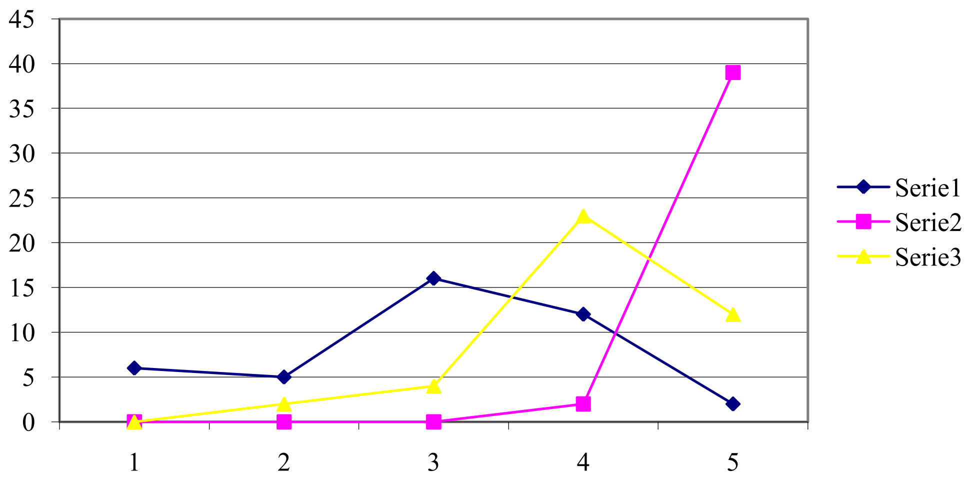

Fig. 2. Survey responses, cont

Source: own research.

1. Utility of references suggested at lecture (4).
2. Punctuality of the instructor (5).
3. Consistency of calculus with the subject of studies (6).

Table 1. Mean, standard deviation and variability coefficient

	1	2	3	4	5	6
Mean	3.04878	3.853659	4	2.97561	4.95122	4.097561
Deviation	0.920631	1.038291	0.948683	1.106522	0.218085	0.768273
Var. coefficient	0.301967	0.26943	0.237171	0.371864	0.044047	0.187495

Source: own elaboration.

The clarity of lectures (content assimilability) and the utility of references are largely dependent on the skill level and students‟ willingness to learn. From the instructor perspective, the awareness of student skill level may result in three strategies: first – reduce the pace to that of the weakest student; second – follow the official calculus recommendations of the Ministry of Education; and third – seek the optimal method of education.

The first of the above strategies does not provide support to the most promising of the students. The second – formal and actually employed – failed to bring results (more than 50% of the students failed the examination). The third strategy requires time – a scarce commodity at present.

Reference suggestions covered a wide range of material, from elementary: A. Ostoja-Ostaszewski: *Matematyka w ekonomii* [*Mathematics in Economics*], T. Bednarski: *Elementy matematyki w naukach ekonomicznych* [*Elements of Mathematics in Economic Sciences*]; through intermediate: Antoniewicz and Misztal: *Matematyka dla studentów ekonomii* [*Mathematics for Students of Economics*]; to fairly advanced: A. Smoluk: *Podstawy analizy matematycznej* [*Fundamentals of Mathematical Analysis*]. For classes, a book of *Zbiór zadań z analizy* [*Analysis problems*] by Krysicki and Włodarski, was recommended. However, as shown both in the survey responses and personal reports obtained during classes and office hours, students do not read the material suggested. Consequently, they do not have a chance to broaden their understanding of the material covered at lectures nor prepare for the classes (designed to build and test their understanding of the lecture material).

3. Developing the ability to think

Passing grade at the examination was set at 30 of 70 points. The test included the following problems (each ranked 0 to 10):

1. Plot the graph of function:

$$f(x) = (x+2) \cdot e^{\frac{1}{x}}.$$

2. Compute the value of improper integral:

$$\int_0^{\infty} \frac{1}{4+9x^2} dx.$$

3. Provide the reverse for transformation:

$$f : R^3 \to R^3$$

$$f(x, y, z) = (x + 2z, x + y + z, -2x + 2y - 5z).$$

4. Solve the following set of equations using Gauss method (matrix reduction):

$$\begin{cases} x - y + 2z = 1 \\ 2x - 2y + 5z = 2 \\ -x + y - 3z = -1 \end{cases}.$$

5. Examine linear independence of vectors:

$$\{I, A, A^2\},$$

where

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}, \quad I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

6. Calculate

$$\iint_D (x-y)^2 dxdy,$$

where D is an area limited by coordinate system ($x = 0$ and $y = 0$) and line $x + y = 1$.

7. A consumer may spend 90 PLN on two articles, each priced 1 PLN a piece. The utility function describing his value preference for x pieces of the first article and y pieces of the second article is expressed by equation

$$u(x, y) = x^{\frac{1}{2}} y^{\frac{1}{3}}.$$

Calculate the maximum utility basked of goods.

The most problematic were exercises 1 and 2 (with only 10% and 15% correct answers, respectively). These two problems require a certain amount of deduction, i.e. ability to combine disparate notions. The remaining exercises tested the straightforward, “mechanical – reconstructive” abilities. The final test results – more than 50% of fail grades – may in fact be regarded as success, in the light of the initial 66% of students that were admitted without the school-leaving maths exam grade (optional, in the light of present legislature), but the reluctance on the part of great majority of students to think and show effort is truly alarming (students were allowed to use their own lecture notes during the examination).

The measurement of education effectiveness typically employs the index of information accessibility, i.e. its relation to age, skill level and development level of the student:

$$D = \frac{w}{d};$$

where d and w represent total didactic information and that portion of information that is converted into knowledge, respectively.

In the light of the above, it can safely be assumed that the index for the lecture course under study was below the 50% margin.

Many thinkers, ethicists, sociologists and researchers often emphasize the argument of "lack of thinking", as one of the most important, and at the same time, neglected and disregarded sources of the problem at hand. In an academic setting, "acceptance for non-thinking" strongly contrasts with the fundamental mission of academy and has a detrimental effect on some of the most basic human needs, such as the need to satisfy curiosity and search for truth, that are based on the fundament of reasoning ability.

The conclusion is straightforward: we need to help students develop the ability to think. The question remains: how?

The starting point – as stated in the first section above – is the knowledge of abilities and capabilities of individual students. Problems and tasks assigned help channel student thinking, but they have to be tailored to the potential of recipients. Tasks should not be banal for the most able students, and not too difficult for those that lack basic knowledge. To make that kind of informed selection, the teacher must know the target group. This precondition is unattainable in the case of large groups and under short time frames (the course under study involved as few as 15 class hours).

Methodology is well-established and known since antiquity: creative thinking is best taught using problem method, with a teacher restraining from offering ready solutions (proofs), while stimulating thought processes through probing questions. Therefore, it seems that the best approach is one based largely on student self-study and individual effort. This, however, is viable only if the student is vitally interested in gaining knowledge and understanding of the material covered in the course of studies. Material provided at lectures must be verified in practice by each individual student, otherwise the recipients fail to gain satisfaction out of apprehended knowledge.

Pope John Paul II in his address to academic community in Częstochowa (1979) made the following observation: *The duty of a university is to teach, but the predominant task of an academic institution is to help young people, those who come with fairly well-established knowledge and experience, gain the ability of independent thinking on their own.* [...] *The academic institution is one of the greatest masterpieces of human culture. But there is a serious fear that this masterpiece faces distortion in this age, and on global scale. I do not know, I am no longer an expert, if I ever was. But I fear the prospect*.

4. Managing the education process

There are many coefficients of the "education production" function that describes teaching process. Below, some of the well-documented facts of this function are presented

Coleman"s report (J.S. Coleman (1966)) suggests that education results depend, in equal measure, on school and on the characteristics of the student and his/her family.

E. Hanushek (1986), in his study on education effects of children and teenagers attending state-run schools in 1980s, arrives at similar conclusions: the level of student upkeep expenses has a markedly lower effect on the student"s achievements than his or her social (familial) background.

Card and Krueger (D.A. Card, A.B. Krueger (1996)) demonstrate that school results directly correlate with parents" salary level, although in some cases this correlation is not strong.

E. Lazear (2001) presents proofs for several interesting theses related to education function:

1. Optimal number of students in a class grows with teacher"s earnings, decreases with the value of teaching unit (on labour market) and increases with the probability of student"s preparation to classes and their discipline at school.

2. Education results are better in large classes consisting of well-prepared and disciplined students, compared with less populated classes of ill-prepared and ill-behaved students.

3. Student achievements reach maximum if the students are sorted according to their abilities and discipline.

E. Hanushek (2003), based on the research of school results in state-owned US schools, demonstrated that the basic education function coefficients such as number of students per teacher, teacher experience, education and salary level, as well as the level of expenditure per student bear little statistical significance.

Table 2. Public schools in the US, 1960-2000

	1960	1970	1980	1990	2000
Students/teacher	25.8	22.3	18.7	17.2	16.0
% of teachers with academic education	23.5	27.5	49.6	53.1	56.2
Mean teacher work experience (in yrs) *i*	11	8	12	15	15
Current expenditure per student $	2.235	3.782	5.124	6.867	7.591

Source: US Department of Education (E. Hanushek (2003)).

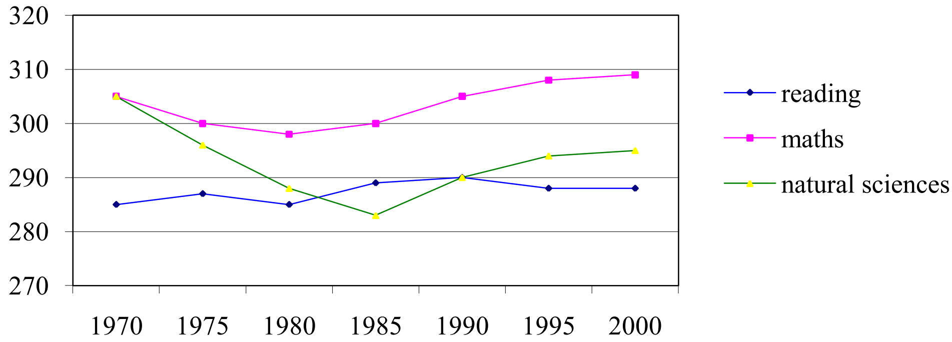

Fig. 3. Results of students aged seventeen in US public schools

Source: US Department of Education (based on school skills test) – E. Hanushek (2003).

Apart from low rate of growth, US schools also face the problem of high student withdrawal rate.

Of ca. 2.4 million US citizens graduating from American public high schools every year, nearly a fourth (600 000) fail to show reading and writing skills of 8th grade level. It must be noted that this gruesome figure does not take into account another million of young people who withdraw before graduation. – America's Schools Still Aren't Making the Grade, "Business Week" September 1988.

It seems that this lack of cohesion between theory and practice results not only from the weak theory, but, most of all, from specificity of the US education sphere. High withdrawal rate observed in US high schools suggests large disparity of knowledge, skills and discipline. Polish situation in this respect is not as pronounced as that reported in the US, but still considerable disparities in knowledge, skill and discipline may be observed, also on academic level. From the 3rd postulate of Lazar, one may safely assume that, given large disparities in skills and discipline in a group of students, maximum education results cannot be achieved. This naturally leads to the thesis: the higher the disparity of knowledge, the worse the results of education.

Based on the above, it seems valid to suggest that education results be related to disparity level and mean level of knowledge among students. This argument may be elaborated using instruments and theorems of welfare economics.

If $U(x)$ is the utility of lecture for a student of knowledge, skill and discipline x, and $f(x)$ represents density of such distribution in a population of students, then the mean utility of lecture (*SUW*) for that group may be calculated using the equation:

$$SUW = \int_0^\infty U(x) f(x) dx. \tag{1}$$

(analogue of social welfare in a population of income distribution $f(x)$).

A.F. Shorrocks (1983), using generalized Lorenz curve $GL(x)$:

$$GL(x) = \mu \cdot L(x), \tag{2}$$

where $L(x)$ is Lorenz curve and μ represents mean income, provided proof for the following theorem:

Theorem. *Let* $F(x)$ *and* $G(x)$ *be distribution function of* 2 *income distributions* ($f(x)$ *and* $g(x)$, *respectively to density*). *Then*

$$GL_F(p) \geq GL_G(p) \Leftrightarrow \int_0^\infty U(x) f(x) dx \geq \int_0^\infty U(x) g(x) dx \tag{3}$$

for any $U(x)$ *that satisfies* $U'(x) > 0$ *and* $U''(x) < 0$, *and for any* $p \in [0, 1]$.

Therefore, SUW for population *A* is equal or higher than SUW for population *B* only if *A* population distribution dominates (in the sense of generalized Lorenz curves) over *B* population distribution.

The generalized Lorenz curve for a population of *F* distribution is defined as follows:

$$GL(F;\ p) = \int_0^p F^{-1}(q) dq\,, \quad \text{for } p \in [0,\ 1], \tag{4}$$

while the correlated partial order *GL* is defined as:

$$FGLG \Leftrightarrow GL(F;\ p) \geq GL(G;\ p)\,, \quad \forall p \in [0,\ 1],$$

and (5)

$$GL(F;\ p) > GL(G;\ p) \quad \text{for a given } p \in [0,\ 1].$$

A.K. Sen (1973) introduced the following reduced measure of welfare (Sen index), that may be used to measure knowledge in a given population:

$$IS = \mu\,(1 - G).$$

Remark 1. Order of the generalized Lorenz curves implies the order of the reduced measure of knowledge (welfare) of Sen:

$$FGLG \Rightarrow \int_0^1 GL_F(p)dp \geq \int_0^1 GL_G(p)dp \Leftrightarrow \mu_F \int_0^1 L_F(p)dp \geq \mu_G \int_0^1 L_G(p)dp \Leftrightarrow$$

$$\Leftrightarrow \mu_F\left(\frac{1-2S_1}{2}\right) \geq \mu_G\left(\frac{1-2S_2}{2}\right) \Leftrightarrow \tfrac{1}{2}\mu_F(1-G_F) \geq \tfrac{1}{2}\mu_G(1-G_G) \Leftrightarrow IS_F \geq IS_G,$$

where S_1 is the area which is upward-limited by Lorenz curve of F, while S_2 is the area which is upward-limited by Lorenz curve of G.

Remark 2. The reverse of the above implication is false, since the order of generalized Lorenz curves is a partial order, while the order described by Sen welfare measure is a linear order.

Therefore, to measure the effectiveness of a given lecture (school), it will suffice to calculate the difference of Sen knowledge measure taken after and before the lecture under study (or at the end and at the beginning of the semester under study).

Another approach is to use Kakwani measure (index) (IK) in place of the Sen index:

$$IK = \frac{\mu}{1+G},$$

where μ represents mean income, and G is the Gini coefficient.

Sen index (IS) depends more on μ than on G for $G \leq 0.5$; while for $G > 0.5$ the reverse holds true, i.e. the level of knowledge for a given population is more dependent on G than on μ. The Kakwani index depends more on μ than on G for all cases.

The effectiveness of the education process may also be perceived through the number of student withdrawals from the course (either intentional or caused by exam failure), or by the number of students graduating in due course. Those measures, however, may provide false description of education effectiveness. To demonstrate this, let us use a simple example (Gori and Vittadini) (Tab. 3).

As seen in the table above, 15% of students graduate from university A, compared with 17% for university B. This may lead to the conclusion that the education effectiveness of university B is higher than that of university A. However, university A shows higher effectiveness in all ranges of points received at school-leaving exam. This measure for university A amounts to: 8%, 16,7% and 30%, compared with: 5%, 10% and 23,3%, respectively, for university B.

Table 3. The risk of simple indexes

Secondary school-leave exam (pts in %)	University A			University B		
	I year students	Graduates	% (Grad/I)	I year students	Graduates	% (Grad/I)
30–52	50	4	8.0	20	1	5.0
53–76	30	5	16.7	20	2	10.0
77–100	20	6	30.0	60	14	23.3
Total	100	15	15.0	100	17	17.0

Source: own elaboration.

The Gini index for the I year students of university A equals 0.57, compared with 0.33 for university B. Sen index of knowledge for A students is 24.61, and 49.36 for university B.

$$\frac{IS(A)}{IS(B)} = \frac{24.61}{49.36} = 0.498 \approx 0.5 . \quad \frac{15}{17} = 0.882 \approx 0.9 .$$

Therefore, knowledge measure ratio (calculated using Sen index) of these two universities equals ½ *at input*), and 0.9 *at output*. Hence, it may safely be assumed that university A offers greater effectiveness of education than university B (nearly twice more effective). Similar results may be observed for any of the point ranges shown. This leads to the conclusion that the postulated measure of effectiveness offers more realistic results compared with the method based on percentage of graduates.

It seems, therefore, that lectures addressed to a widely-disparate group of students (in terms of their knowledge level) are not an effective solution – or, at least, sub-optimal.

The closedown of vocational schools without proper examination of real demands of the labour market and without any attempts to remedy calculus and quality of education, coupled by the unification of the whole education system (universal schools of secondary level plus universities) will undoubtedly lead to prolonging the time of learning, but does it offer more intensive education? The above considerations suggest that such an approach will not result in optimization. On the contrary – in effect, education effectiveness will drop and intellectual potential of students and pupils will go to waste.

At this point, one may ask: what are the reasons for adopting such an approach to education? The answer is simple and clear in the light of the following data:

Table 4. Comparison of education expenditures and education results in selected countries

	A	B	C	D	D/B (%)
United States	24 074	40 088	39	490	1.22
Canada	19 992	19 994	31	526	2.63
France	10 704	33 548	53	506	1.50
Germany	11 594	48 167	64	499	1.03
Italy	8 764	31 291	53	476	1.52
Czech Republic	6 774	37 925	59	509	1.34
Poland	4 589	10 263	15	495	4.82

Source: OECD in Figures, 2007 Edition.

where:
A – annual expenditures per university student in $ based on PPP,
B – annual salary of a secondary school teacher with 15-year practice in $ based on PPP,
C – wage per hour of didactic work of such a teacher,
D – mean knowledge and skill coefficient of a 15-year-old pupil.

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