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SPECTRAL DENSITY TESTS IN VAR FAILURE CORRELATION ANALYSIS

Summary: The paper presents application of the spectral theory, developed primarily in physics, to risk analysis in economics. The aim of the paper was to evaluate statistical properties of spectral density-based VaR tests based on various test statistics and to compare them to the popular Christoffersen's Markov test. Test assessment included their size and power. The analysis of the test properties was preceded by the overview of spectral theory methods proposed in the literature for testing VaR failure correlation. The statistical properties of the considered tests were evaluated through the Monte Carlo method. The study showed that the spectral test approach outperformed the Markov test in terms of the test power, however, the asymptotic distributions of the test statistics did not ensure the proper size of the tests. The comparison of the four considered spectral test statistics indicated the superiority of the test based on the Cramer-von-Mises statistic over all other tests.

Keywords: Spectral analysis, correlation theory, spectral density test, VaR test.

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1. Introduction

Developed primarily as means to describe physical phenomena like the vibrating string or heat flow, the Fourier analysis can also be applied in economics to facilitate statistical signal processing. The power spectrum, obtained from the time series as a Fourier transform of the autocovariance function gives detailed information about the time structure of the process. The information content of the power spectrum corresponds to the autocovariance, which, in case of normally distributed data, together with the mean, completely specifies the joint probability distribution of the data. Only if the distribution is far from normal, might it become interesting to study higher order moments or other characteristics of the process. The power spectrum and the autocovariance function are equivalent, which means that if either is known, the other can be determined exactly. However, these functions display different aspects of the correlation information about the time series. The power spectrum quantifies the total power, which is the measure of the total activity or volatility of the process and exhibits variations of power with frequency. In economics, the spectral analysis may reveal hidden periodicities in the studied data, resulting from the cyclic behaviour of the process. Thus for practical purposes the spectrum might be the more useful parameter than the autocovariance function itself.

Historically, the introduction of spectral methods for time series followed two lines of development. The first line, originating in the study of light in physics, covered the spectral analysis for functions with finite power [Wiener 1930]. Wiener's theory explored both univariate and multivariate time series, and applied to stochastic as well as nonstochastic series. The second line of development included both stationary and weakly stationary stochastic processes and delivered the fundamental theorems of the correlation theory for weakly stationary processes [Khintchine 1934]. This line of enquiry was one of the pioneering studies in the modern theory of stochastic processes. Later the theory of weakly stationary time series was investigated in the context of the energy analysis in turbulent flows [Kołmogorov 1941]. The crucial advancement was spectral decomposition of weakly stationary processes, based on the Fourier transform of the autocovariance function [Cramer 1942].

In economics the spectral decomposition of the stochastic processes was employed in the correlation analysis. The idea of the spectral density was researched both in context of serial correlation in univariate time series and in analysis of correlation between separate economic processes [Talaga, Zieliński 1986]. Characterized by the spectral density function, the power spectrum of the martingale difference process was discovered to be a flat line. This development led to the idea of the martingale property test based on the measure of distance between the theoretical flat line and the observed spectral density of the process [Durlauf 1991]. Spectral density tests have power against linear alternative of any order, therefore it was proposed to apply the spectral approach in market risk analysis to test the correlation in the VaR failure series [Berkowitz, Christoffersen, Pelletier 2011].

The paper explores the application of spectral methods in the market risk analysis. The aim of the paper was to conduct a comparative study of spectral tests aimed at detecting correlation in VaR failure series. The popular Markov VaR correlation test was used as benchmark [Christoffersen 1998]. The comparative analysis was preceded by the comprehensive overview of spectral methods in context of time series correlation analysis. Statistical properties of the tests were evaluated through the Monte Carlo study, which included the size and the power of the tests.

2. Spectral methods in correlation analysis

2.1. Fourier transform of function. Three representations

Historically, the development of the Fourier analysis was motivated by the studies of the physical phenomena. The laws governing these phenomena were expressed by partial differential equations, like the wave equation, which were solved in terms of Fourier series. The solution of these differential equations leads to the wide class of periodic functions, obtained by composing a number of simple harmonic functions with various coefficients

$$y(k) = \sum_{\omega>0} a_\omega \cos \omega k + b_\omega \sin \omega k, \quad k \in R. \tag{1}$$

In the formula (1) the symbol $\Sigma_{\omega>0}$ represents any number of possible summing operations, including integrals (see e.g. [Koopmans 1995, p. 9]). In terms of the Fourier analysis, the representation (1) is the inverse Fourier transform of the function $y : R \to R$ (the detailed overview of the Fourier analysis can be found e.g. in [Stein, Shakarchi 2003]). In economics $y(k)$, $k \in R$ may be interpreted as the autocovariance of order k of the stochastic process and the spectral representation (1) may be used to identify the cyclic behaviour of the process.

Through the trigonometric identity $\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$ the Cartesian representation (1) can be transformed into the alternative form

$$y(k) = \sum_{\omega>0} A_\omega (\sin \omega k + \phi_\omega), \quad k \in R. \tag{2}$$

For each $\omega > 0$ the nonnegative parameter $A_\omega \geq 0$ is the maximum amplitude of the process and the ϕ_ω parameter denotes the phase, which measures the displacement of the sinusoid relative to the given time origin. Because of the periodic repetition of the sinusoid, the phase can be restricted to the range $-\pi < \phi_\omega \leq \pi$, $\omega > 0$. The amplitude and phase can be recovered from a_ω and b_ω by the relations $A_\omega = \sqrt{a_\omega^2 + b_\omega^2}$ and $\phi_\omega = arctg(\frac{b_\omega}{a_\omega})$ (see e.g. [Stein, Shakarchi 2003, p. 3]).

Another representation of (1) is based on the identity $\sin\alpha = \frac{e^{i\alpha} - e^{-i\alpha}}{2i}$, which is a consequence of the de Moivre relation $e^{i\alpha} = \cos\alpha + i\sin\alpha$. By simple transformations we can arrive at $A_\omega(\sin\omega k + \phi_\omega) = c_\omega e^{i\omega k} + c_\omega e^{i(-\omega)k}$, where $c_0 = A_0$ and $c_\omega = \frac{A_\omega e^{i\phi_\omega}}{2i}$. Thus, if we adopt the notation $c_{-\omega} = c_\omega$ equation (2) can be written in the form

$$y(k) = \sum_{\omega \in R} c_\omega e^{i\omega k}, \quad k \in R, \tag{3}$$

which is called the complex representation of (1). The sum in (3) is taken over both positive and negative frequencies, which shows that if (2) has a harmonic component with frequency ω, the complex representation (3) has two harmonic components, one with frequency ω and the other with frequency $-\omega$ (see e.g. [Stein, Shakarchi 2003, p. 15]).

The description of the cyclic behaviour of the stochastic process requires identification of the coefficients c_ω, $\omega \in R$ in (3). This is based on the theory of the correlation function and its spectral representation. The central point in this theory is the Fourier transformation used to convert the autocovariance function to obtain the spectral distribution.

2.2. Spectral representation of autocorrelation function

Let $x(t)$, $t \in R$ represent a real-valued realizations of the stochastic process with the property that autocovariance function

$$\operatorname{cov}(k) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} x(t)x(t+k)dt, \tag{4}$$

exists and is finite for every $k \in R$. Wiener established the existence of a bounded nondecreasing function F, called the spectral distribution function, such that

$$\operatorname{cov}(k) = \int_{-\infty}^{\infty} e^{i\omega k} dF(\omega), \quad k \in R. \tag{5}$$

Expression (5) is called the spectral representation of the autocovariance function. The spectral distribution function determines a measure F called the spectral distribution of the time series and (5) is an integral with respect to this measure. $F(A)$ can be interpreted as the amount of power in the harmonic components of the time series with frequencies in the set A. The spectral distribution and spectral distribution function are related by the expression

$$F(\omega) = F((-\infty, \omega]), \quad \omega \in R \tag{6}$$

(see e.g. [Koopmans 1995, p. 31]).

For most practical situations the integral (5) can be reduced to the discrete and continuous components by the Lebesgue decomposition $F(A) = F_d(A) + F_c(A)$ (see e.g. [Grenander, Rosenblatt 1957, p. 35]). The discrete spectral distribution $F_d(A)$ is characterized by a spectral function $p : R \to R$, satisfying $p(\omega) \geq 0$ for all ω in R and $p(\omega) > 0$ only in a discrete set of frequencies $B = \{\ldots\omega_{-1}, \omega_0, \omega_1, \ldots\}$, where $\omega_{-j} = -\omega_j$ (thus $\omega_0 = 0$). The value of the spectral function $p(\omega)$ at frequency ω is determined directly by the spectral distribution F and results from the relation $p(\omega) = \mathrm{F}(\{\omega\})$, $\omega \in B$. The continuous component $F_c(A)$ is characterized by the spectral density, which can be computed as the derivative of the spectral distribution function $f(\omega) = F'(\omega)$ in all continuity points of F. The discrete and continuous components of the spectrum determine the spectral mass or power in a set of frequencies A in the following way:

$F(A)=\sum_{\omega_j\in A}p(\omega_j)+\int_A f(\omega)d\omega.$ The spectral representation of the autocovariance (4) can then be written in the form

$$\operatorname{cov}(k)=\sum_{j=-\infty}^{\infty}e^{i\omega_j k}p(\omega_j)+\int_{-\infty}^{\infty}e^{i\omega k}f(\omega)d\omega,\quad k\in R,\tag{7}$$

which is analogous to formula (3) (see e.g. [Koopmans 1995, p. 32]).

The fundamental question of obtaining the spectral representation refers to the Fourier transformation. When the time series has a discrete (or point) spectrum, i.e. when $f(\omega)=0$ for all ω, then

$$\operatorname{cov}(k)=\sum_{j=-\infty}^{\infty}e^{i\omega_j k}p(\omega_j),\quad k\in R.\tag{8}$$

The spectral function, called also the spectrum of the series, can be computed through the discrete Fourier transform of the autocovariance function, which has the form

$$p(\omega)=\lim_{t\to\infty}\frac{1}{2\pi}\int_{-T}^{T}\operatorname{cov}(k)e^{-i\omega k}dk,\quad \omega\in R.\tag{9}$$

When the spectrum has only the continuous component, the covariance function is represented by

$$cov(k)=\int_{-\infty}^{\infty}e^{i\omega k}f(\omega)d\omega,\quad k\in R.\tag{10}$$

Then $f(\omega)$ can be obtained by

$$f(\omega)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\operatorname{cov}(k)e^{-i\omega k}dk,\quad \omega\in R,\tag{11}$$

which is the continuous Fourier transform of the autocovariance function. Expressions (10) and (11) are called the Wiener-Khintchine relations. In some situations it may not be possible to obtain the spectrum from cov(k), however this is regarded as rather pathological in practice. The Wiener-Khintchine theorem [Wiener 1930; Khintchine 1934] defines conditions for autocovariances to have a Fourier transform that is nonnegative everywhere (see e.g. [Broersen 2006]).

The spectral representation of the autocovariance function determines the total power of the time series, which is a measure of the total activity or volatility of the process. It also represents the power decomposition over frequencies. By definition the total power is $P(x(t))=\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}x^2(t)dt,$ hence $P(x(t))=cov(0).$ Substituting $h=0$ in (7) gives

$$P(x(t)) = \text{cov}(0) = \sum_{j=-\infty}^{\infty} p(\omega_j) + \int_{-\infty}^{\infty} f(\omega)d\omega. \tag{12}$$

This decomposition shows that the power in the discrete and continuous components contributes additively to the total power of the time series, and each of these is the sum of the power magnitudes at each frequency (see e.g. [Stoica, Moses 2005, p. 7]).

In practice the estimator of the discrete spectrum is based on its sample equivalent called the correlogram

$$\hat{p}_c(\omega) = \sum_{k=-(T-1)}^{T-1} \text{cov}(k)e^{i\omega k}, \quad \omega \in R, \tag{13}$$

where cov(k) denotes an estimate of the covariance of order k, obtained from the available sample of T observations. Another estimator of the power spectrum, called periodogram, is based on a different representation of the discrete spectrum, which for the discrete time is given by

$$p(\omega) = \lim_{T\to\infty} E(\frac{1}{T} | \sum_{t=1}^{T} x(t)e^{-i\omega t} |^2), \quad \omega \in R. \tag{14}$$

The periodogram is

$$\hat{p}_p(\omega) = \frac{1}{T} | \sum_{t=1}^{T} x(t)e^{-i\omega t} |^2 . \tag{15}$$

When cov(k) is evaluated using the standard biased estimator

$$\text{cov}(k) = \frac{1}{T} \sum_{t=k+1}^{T} x(t)x(t-k), \quad 0 \le k \le T-1,$$

the estimator of the power spectrum $\hat{p}_c(\omega)$ coincides with $\hat{p}_p(\omega)$ (see e.g. [Stoica, Moses 2005, p. 24, 25]).

3. Spectral VaR correlation tests

3.1. Concept of the test

The spectral theory offers methods that may be employed in market risk analysis to test for the correlation in value-at-risk (VaR) failure series. Traditionally, VaR tests were based on direct exploration of the properties of VaR failure process defined as

$$I_t = 1_{\{-x_t > \text{VaR}_p(x_t)\}}, \tag{16}$$

Where x_t represents the rate of return from investment at time t and $\text{VaR}_p(x_t)$ is the value at risk of x_t at time t, $t = 1, \ldots, T$ on the level of tolerance p, $p \in (0,1)$.[1] The idea behind application of the spectral theory in market risk management is to construct a VaR correlation test based on the shape of the spectral density function, which is the Fourier transform of the autcovariance function of the failure process $(I_t)_{t=0}^{T}$.

Let us consider the hypothesis that the process $(I_t)_{t=0}^{T}$ is the martingale difference sequence $H_0 : E(I_t \mid \Omega_{t-1}) = p$, where Ω_t is the information set available at time t and $p \in (0,1)$ is the fixed tolerance level. The equivalent statement is that autocorrelation function $\rho_I(k)$ and autocovariance function $\sigma_I(k)$ of the random variable I_t equal zero for all lag orders k, $k \in Z$ [Berkowitz, Christoffersen, Pelletier 2011].

The test works on the idea of comparing the spectral density function f_I, which is the Fourier transform of the autocovariance function $\sigma_I(k)$,

$$f_I(\omega) = \frac{1}{2\Pi} \sum_{k=-\infty}^{\infty} \sigma_I(k) e^{-ik\omega} \tag{17}$$

to the theoretical spectral density of the martingale difference process. Computation of the spectral density under H_0 falls back on the asymptotic properties of the sample autocorrelation function. Under H_0 it holds that:[2]

1. $\hat{\rho}_I(k) \xrightarrow{d} 0$,
2. $\sqrt{T}\hat{\rho} \xrightarrow{d} N(0, I)$,

where I is the identity matrix of order K, $\hat{\rho}$ is the K-element autocorrelation vector and $\xrightarrow{d}$ denotes the stochastic convergence [Durlauf 1991].

The above properties imply that, under H_0, the spectral density is a flat function

$$f_I(\omega) = \frac{\sigma_I(0)}{2\Pi} = \frac{p(1-p)}{2\Pi}. \tag{18}$$

Thus the relevant cumulative distribution function is given by

$$F_I(\lambda) = \int_{-\Pi}^{\lambda} f_I(\omega) d\omega = \frac{\sigma_I(0)}{2\Pi}(\lambda + \Pi) = \frac{p(1-p)}{2\Pi}(\lambda + \Pi). \tag{19}$$

[1] For an overview of VaR theory see e.g. [Christoffersen 2012]. The comparative study of the basic VaR tests can be found in [Małecka 2013].

[2] The asymptotic properties of the sample autocorrelation are established under general regularity conditions about the moments of the examined stochastic process. These conditions can be found in [Durlauf 1991].

3.2. Test statistic

Verification of H_0 requires investigating the convergence of the spectral density estimator and the measure of the distance between the observed and the theoretical spectral density function. The spectral density (17) is estimated by

$$P_T(\omega) = \frac{1}{2\Pi} \sum_{k=-(T-1)}^{T-1} \hat{\sigma}_I(k) e^{-ik\omega}. \tag{20}$$

The distance between the estimated spectral density and the theoretical flat line is measured by

$$\gamma_T(\omega) = \frac{1}{2\Pi} \left(\sum_{k=-(T-1)}^{T-1} \hat{\sigma}_I(k) e^{-ik\omega} - \sigma_I(0) \right). \tag{21}$$

The above formula represents a random function, hence it cannot be used directly as a test statistic. This function does not exhibit point convergence. Under H_0 the cumulated function converges to zero, which allows for measuring the discrepancy between the observed series and the H_0 by the function

$$\Gamma_T(\lambda) = \int_0^\lambda \left(P_T(\omega) - \frac{\hat{\sigma}_I(0)}{2\pi} \right) d\omega, \quad \lambda \in [0, \Pi]. \tag{22}$$

The test statistic is based on the modification of the formula (22)

$$U_T(t) = \sqrt{2T} \int_0^{\Pi T} \left(\frac{P_T(\omega)}{\hat{\sigma}_I(0)} - \frac{1}{2\pi} \right) d\omega = \tag{23}$$

$$= \frac{\sqrt{2}}{\Pi} \sum_{h=1}^{T-1} \sqrt{T} \hat{\rho}_I(h) \frac{\sin h\Pi t}{h}, \quad t \in [0,1],$$

in which λ is converted to Πt, $t \in [0,1]$, the multiplication by $\sqrt{2}$ is used to facilitate computation of the asymptotic distribution[3] and the normalization factor $\frac{\sigma_I(0)}{\sqrt{T}}$ is introduced. That gives the formula being the function of the sample autocorrelation normalized by $\sqrt{T}$. In opposite to the expression (22), the modified statistic is robust against heteroskedasticity [Durlauf 1991].

Under H_0 it holds that $U_T(t) \xrightarrow{d} B(t)$, $t \in [0,1]$ where $B(t)$ is the Brownian bridge on $[0,1]$.[4] Using the above convergence the martingale property test can be

[3] The proof can be found in [Durlauf 1991].

[4] The Brownian bridge is the Wiener process conditional on $W(1) = 0$, therefore $B(t) = W(t)|\{W(1) = 0\}$, $t \in [0,1]$.

conducted through a number of test statistics, which map the random function $U_T(t)$ into the random variable. The following statistics are proposed: Anderson-Darling statistic

$$SD_{AD} = \int_0^1 \frac{U(t)^2}{t(1-t)}, \tag{24}$$

Cramer-von-Mises statistic

$$SD_{CVM} = \int_0^1 U(t)^2, \tag{25}$$

Kolmogorov-Smirnov statistic

$$SD_{KS} = \sup_{t\in[0,1]} |U(t)| \tag{26}$$

and Kuiper statistic

$$SD_{Kui} = \sup_{0\le s,t\le 1} |U(t) - U(s)| \tag{27}$$

[Deo 2000].

4. Comparative study of test properties

4.1. Size evaluation

The statistical properties of the test statistics based on spectral density function and the LR_{ind} statistic of the Christoffersen's Markov test, used as a benchmark VaR test for its popularity, were compared through the simulation study. The size and power were estimated as the proportion of rejections under the null (type-one errors) and the proportion of rejections under the alternative respectively. The estimates were computed over 10000 replications for sample sizes T = 250, 500, 750, 1000. The level of significance was set to 5%. The size assessment was done through generating i.i.d. Bernoulli samples with the probability p = 0.05, equal to the chosen VaR tolerance level. The size evaluation was complemented by the graphical analysis, where the empirical and theoretical distribution functions of the test statistics were compared.

The comparison of the empirical and theoretical distributions of all considered test statistics – LR_{ind}, SD_{KS}, SD_{Kui}, SD_{CVM}, SD_{AD} – showed that the empirical size of the Markov LR_{ind} test was further from the nominal test size than the size results obtained for spectral tests and there was the opposite direction of the divergence (Table 1). In contrast to spectral tests the empirical size of the LR_{ind} test was higher than the nominal level of 5%, hence the asymptotic distribution function was moved

Table 1. Size estimates of the spectral VaR tests

Test	Series length			
	250	500	750	1000
LR_{ind}	0.071	0.083	0.122	0.137
SD_{KS}	0.016	0.029	0.033	0.032
SD_{Kui}	0.019	0.025	0.027	0.027
SD_{CVM}	0.033	0.028	0.031	0.037
SD_{AD}	0.024	0.033	0.037	0.033

Source:xxx.

to the left of the empirical shape (Figure 1). Moreover the obtained results showed no convergence to the nominal level with lengthening the time series.

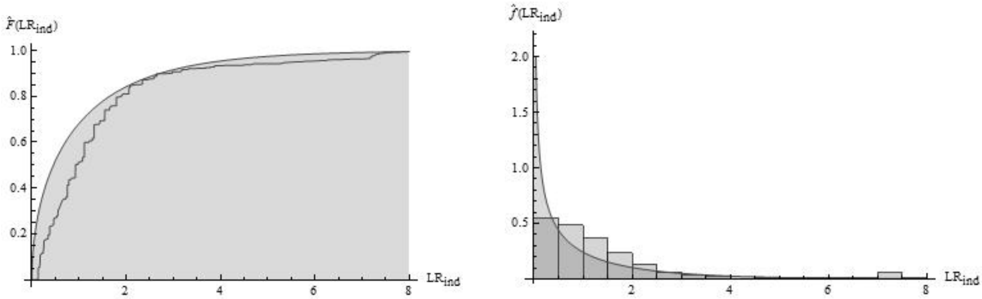

Figure 1. Empirical and asymptotic distribution of LR_{ind} 250 observations

Source: own elaboration.

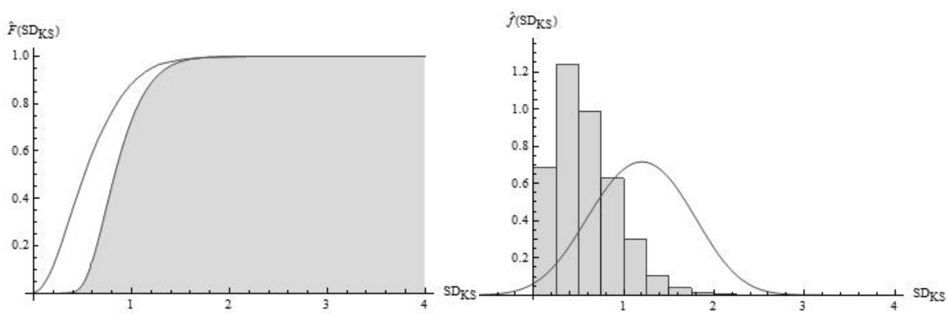

Figure 2. Empirical and asymptotic distribution of SD_{KS} 250 observations

Source: own elaboration.

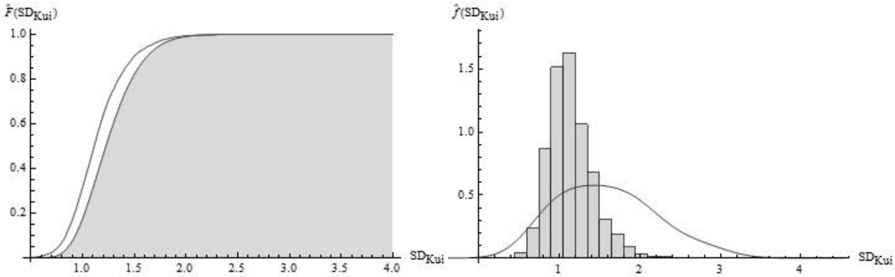

Figure 3. Empirical and asymptotic distribution of SD_{Kui} 250 observations

Source: own elaboration.

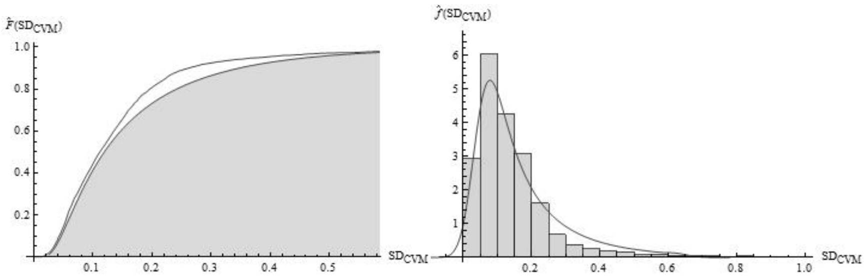

Figure 4. Empirical and asymptotic distribution of SD_{CVM} 250 observations

Source: own elaboration.

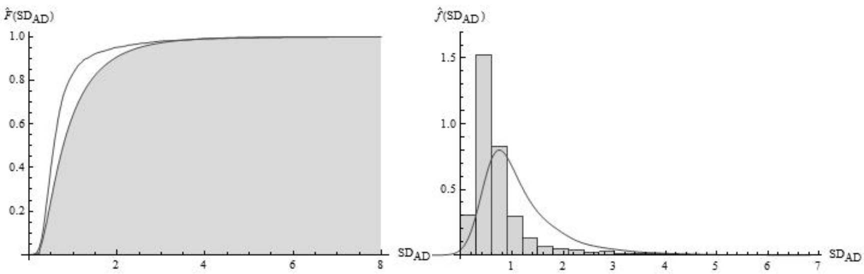

Figure 5. Empirical and asymptotic distribution of SD_{AD} 250 observations

Source: own elaboration.

In the group of spectral tests the asymptotic curve lied to the right of the empirical shape, which indicated the conservative character of the tests (Figures 2–5). The displacement of the distributions resulted in the size estimates below 4%. The results suggested therefore that there might be high probability that the spectral VaR tests applied with the asymptotic distributions do not have power against the possible serial dependence in the time series. The largest compliance between the empirical and asymptotic distribution was observed in case of the SD_{CVM} test, based on Cramer-von-Mises statistic.

4.2. Power evaluation

The power assessment required generating serially dependent failure process, which was done through reference to the Markov property, characterizing memoryless discrete-time processes. The serial dependence was hence represented through the correlation of process values in adjacent periods. The simulation experiment was based on the two-state Markov chain model in which 1 represented the VaR failure and 0 – its non-occurence. The starting distribution (π_0, π_1) was defined by the assumption the probability of failure equalled the chosen tolerance level $\pi_1 = 0.05$ The compliance with the alternative hypothesis was guaranteed by choosing the transition matrix

$$\begin{bmatrix} \pi_{00} & \pi_{01} \\ \pi_{10} & \pi_{11} \end{bmatrix},$$

in which $\pi_{01} \neq \pi_{11}$. The π_{11} parameter was set subsequently to values 0.1, 0.2, 0.3. The respective values of π_{01} were implied by the properties of the probability and π_{00}, π_{01} were fixed at levels guaranteeing stationarity of the process. Hence the experiment of the model ensured the correct overall frequency of failures and their serial dependence.

The power comparison was conducted with the use of the Monte Carlo test technique, which is based on the assumption that the finite sample distribution of the test statistic can be simulated [Dufour 2006]. According to the Monte Carlo test algorithm we used the test statistic distributions simulated under H_0, which ensured the exact test size and guaranteed the comparability of the power estimates.

The results of the power comparison showed the superiority of the spectral VaR test to the LR_{ind} test, based on checking the Markov property in the VaR failure series (Table 2). The observed power of the Markov test was lower than the relevant power results obtained for all considered spectral test statistics.

The spectral test power analysis showed that in cases when the examined process was characterized by the conditional probability of failure $\pi_{11} = 0.3$ or higher the serial dependence was detected in more than 50% of the repetitions, even for the shortest series length. The transition probability $\pi_{11} = 0.5$ gave power estimates higher than 70%, independent of the test statistic and the number of observations. The study showed fast growth in the test power with lengthening the time series, especially with shift from 250 to 500 observations. For 500 observations and serial

Table 2. Power estimates of the spectral VaR tests

Test	π_{11}	Series length			
		250	500	750	1000
LR_{ind}	0.1	0.10	0.13	0.17	0.19
	0.3	0.36	0.59	0.76	0.86
	0.5	0.64	0.89	0.97	0.99
SD_{KS}	0.1	0.18	0.27	0.35	0.43
	0.3	0.56	0.78	0.88	0.97
	0.5	0.81	0.95	0.99	1.00
SD_{Kui}	0.1	0.11	0.14	0.20	0.26
	0.3	0.44	0.66	0.80	0.92
	0.5	0.72	0.89	0.98	0.99
SD_{CVM}	0.1	0.19	0.27	0.33	0.42
	0.3	0.58	0.79	0.88	0.97
	0.5	0.83	0.96	0.99	1.00
SD_{AD}	0.1	0.15	0.28	0.33	0.38
	0.3	0.53	0.79	0.89	0.96
	0.5	0.80	0.95	0.99	1.00

Source: own elaboration.

dependence characterized by π_{11} = 0.3 or π_{11} = 0.5 the estimated power exceeded 60% in case of all tests. The comparative analysis of the power estimates of the four considered spectral test statistics showed the superiority of the SD_{CVM} test which also outperformed other tests in the size exercise.

5. Summary and conclusions

The paper explored the application of the spectral theory in the market risk analysis in economics. Spectral methods were employed as means to verify statistical hypothesis about the correlation in VaR failure series. The empirical study was preceded by the overview of the spectral analysis and the theory of the spectral VaR tests. The testing procedure involved working on the martingale difference hypothesis. The paper presented possible test statistics aimed at measuring the distance between the observed spectral density and the theoretical flat line characterizing the martingale difference property.

The empirical part of the paper presented the analysis of the VaR spectral test properties in comparison to the Christoffersen's VaR test based on the Markov property. The study involved size and power of the test. The properties of the tests were assessed through the Monte Carlo study with the use of the Monte Carlo test technique.

While the size results were not satisfactory for all considered test statistics, the study showed that the spectral test approach outperformed the Markov test in terms of the test power. The size exercise performed for the spectral test showed the discrepancy between the empirical and the asymptotic test statistic distributions, which resulted in the estimated test sizes below nominal significance level. That indicated that the use of the theoretical asymptotic distributions may result in the incorrect tendency to interpret the test outcomes in favour of the zero hypothesis. Hence the power comparison was conducted independent of the asymptotic distributions, relying only of the simulated critical values.

The observed power of spectral tests was higher for all test statistics than the relevant power results obtained the Markov test. In the group of the spectral tests the rejection frequencies under the alternative hypothesis showed fast growth in the test power with lengthening the time series, especially with shift from 250 to 500 observations. The comparison of the four considered test statistics indicated the superiority of test based on the Cramer-von-Mises statistic over all other tests, both in terms of the size and the power.

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TESTY GĘSTOŚCI SPEKTRALNEJ W ANALIZIE KORELACJI PRZEKROCZEŃ VAR

Streszczenie: Praca prezentuje zastosowanie teorii spektralnej, pierwotnie rozwijanej w naukach fizycznych, do analizy ryzyka w ekonomii. Celem pracy była ocena własności statystycznych testów VaR opartych na gęstości spektralnej, wykorzystujących różne statystyki testowe oraz porównanie ich do popularnie stosowanego testu Christoffersena, opartego na łańcuchu Markowa. Badanie własności statystycznych objęło rozmiar i moc testów. Ocena testów poprzedzona została przeglądem metod spektralnych zaproponowanych w literaturze do testowania korelacji wyjątków VaR. Własności statystyczne badano za pomocą metody Monte Carlo. Wyniki pokazały, że w podejściu opartym na teorii spektralnej otrzymano wyższe oceny mocy niż w teście Markowa, jednak uzyskane rozkłady asymptotyczne nie dały gwarancji zachowania odpowiedniego rozmiaru testów. Porównanie czterech testów spektralnych wskazało, że najlepszymi wynikami charakteryzował się test oparty na statystyce Cramera–von Misesa.

Słowa kluczowe: analiza spektralna, teoria korelacji, test gęstości spektralnej, test VaR.