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ON BAYESIAN TESTS IN AUDITING

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Abstract
In auditing the problem of testing hypotheses about frequency of incorrect items is considered. It is treated as the particular case of compliance testing problems. Usually, classical statistical tests are used to testing those types of hypotheses. In the paper the Bayesian approach will be considered. The hypothesis will be tested on the basis of the simple random sample or on the basis of the simple random sample drawn from strata. Usually, Bayesian statistical inference in auditing is based on confidence intervals. Here, instead of that two well known Bayesian rules will be considered. Presented procedures will be illustrated by means of empirical examples.

***Key words**: Bayesian testing of hypothesis, compliance test, stratified population, risk function, Bayes factor.*

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1. Introduction

The quality internal control system is audited. The audit is based on testing methods of processing selected operations by the system under control. The system is good when it has been operating continuously and effectively. Formally, the system is treated as a population of subsystems, which are mutually independent. Let p be probability that the controlled subsystem is wrong. It means that $P(X=1)=p$, $P(X=0)=1-p$. The following hypotheses are considered:

$$H_0: p=p_0, \qquad H_1: p=p_1> p_0,$$

where p_0 is the admissible (tolerable, acceptable) level of the probability that the audited system works incorrectly and p_1 is the inadmissible level of this probability. Moreover, let $p_1>p_0$. Defined hypotheses are usually verified on the basis of the simple sample denoted by $X_1, X_2,\dots, X_n$ where $P(X_i=x)=P(X=x), i=1,..,n,$ while X_i and X_j are mutually independent. It is well known that the sum $M=\sum_{i=1}^{n} X_i$ has Bernoulli distribution with the following probability function.

$$P(M=m \mid p)=\binom{n}{m} p^m (1-p)^{n-m} \qquad (2)$$

According to the classical Neyman-Pearson framework the above hypotheses are usually verified on the basis of the test statistic M or on the basis of its standardized version

$$T = \frac{M - E(M)}{D(M)} = \frac{M - np}{\sqrt{np(1-p)}}$$

So, the hypothesis H_0 is rejected when the p-value evaluated on the basis of formula $\alpha_s = P(M \geq m|H_0) = P(T \geq t|H_0) \leq \alpha$ where α is the assumed significance level. In practice of auditing the probability p is usually very small ($p<0.04$) and in this case there are troubles with the exact evaluation of the probabilities $P(M \geq m|H_0)$ or $P(T \geq t|H_0)$ even when it is assumed that the sample size is large, see e.g. Ryan (2013).

When we allow that the probability p is a value of a random variable, the Bayesian approach is considered to testing the above formulated hypotheses, see e.g. Ghosh and Meeden (1997), Robert (2007), Santer and Duffy (1989) or Statistical Models and Analysis in Auditing (1989). In this case the hypotheses are formulated as follows

$$H_0: p \leq p_1, \qquad H_1: p > p_1, \qquad (1)$$

Now let us underline that p is treated as a value of a random variable and p_1 is fixed. The distribution of p is called a prior one. The framework of Bayesian inference is based on the posterior distribution of p. It is explained in details below under the additional assumptions.

2. Inference under Bayesian approach

2.1 . The homogenous population

Let us take into account the Bayesian model proposed by Meeden (2003). The particular case of that model is defined by the following assumptions. Similarly, like it is above the sum $M = \sum_{i=1}^{n} X_i$ has Bernoulli distribution. The beta distribution $B(u,z)$ is the prior distribution of the probability p with the following density function:

$$f(p) = \frac{\Gamma(u+z)}{\Gamma(u)\Gamma(z)} p^{u-1}(1-p)^{z-1}, \qquad \text{for } p \in (0,1) \qquad (3)$$

where: $\Gamma(\tau) = \int_0^{\infty} x^{\tau-1} e^{-x} dx$, $\tau>0$.

Moreover:

$$E(p) = \frac{u}{u+z}, \qquad V(p) = \frac{uz}{(u+z)^2(u+z+1)},$$

$$\beta_1(p) = \frac{2(u+z)\sqrt{u+z+1}}{(u+z+2)\sqrt{uz}}$$ is skewness coefficient,

The beta distribution is usually taken into account as the prior distribution of the probability p, see, e.g. Santer and Duffy (1989).

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The joint distribution of M and p is:

$$h(p,m)=\frac{\Gamma(u+z)}{\Gamma(u)\Gamma(z)}\binom{n}{m}p^{m+u-1}(1-p)^{n-m+z-1}$$

The marginal distribution of M is:

$$g(m)=\int_0^1 h(p,m)dp=\frac{\Gamma(u+z)}{\Gamma(u)\Gamma(z)}\binom{n}{m}\frac{\Gamma(m+u)\Gamma(n-m+z)}{\Gamma(n+u+z)}$$

The posterior distribution of parameter p is:

$$h(p\,|\,m)=\frac{\Gamma(n+u+z)}{\Gamma(m+u)\Gamma(n-m+z)}p^{m+u-1}(1-p)^{n-m+z-1} \tag{4}$$

Let c_0 be the loss dealing with situation when H_0 is accepted when H_1 is true. It means that the auditor accepts the system when it works incorrectly. Let c_1 be the loss generated by rejecting H_0 when it is true. It means that the well working system is not accepted.

According to the general Bayessian rule of testing statistical hypothesis, see e.g. Krzyśko (2004), pp. 254-5 and 323-5 or Robert (2007), pp. 225-8, the following posterior probabilities are evaluated:

$$P(p<p_1\,|\,m)=\int_0^{p_1} h(p\,|\,m)dp, \qquad P(p\ge p_1\,|\,m)=\int_{p_1}^{1} h(p\,|\,m)dp.$$

So, $c_1P(p<p_1\,|\,m)$ is the risk of accepting the hypothesis H_1 when H_0 is true. Moreover, $c_0P(p\ge p_1\,|\,m)$ is the risk of accepting H_0 when H_1 is true. The decision rule is as follows. The hypothesis

H_0 is rejected when $c_1P(p<p_1\,|\,m)<c_0P(p\ge p_1\,|\,m)$,

H_0 is accepted when $c_1P(p<p_1\,|\,m)\ge c_0P(p\ge p_1\,|\,m)$.

The just written decision rule is equivalent to the following. The hypothesis H_0 is rejected when

$$P(p<p_1\,|\,m)\le\frac{c_0}{c_0+c_1}=r, \tag{5}$$

the hypothesis H_0 is accepted when

$$P(p<p_1\,|\,m)>\frac{c_0}{c_0+c_1}=r. \tag{6}$$

Let us note that if $c_0=c_1$, then $r=\frac{c_0}{c_0+c_1}=\frac{1}{2}$. The value $r=\frac{1}{2}$ is usually taken into account in practical analysis.

The next rule of making decision is based on the basis of the following Bayes factor, see, e.g. Robert (2007), pp. 227:

$$B=\frac{P(p<p_1\,|\,m)}{P(p\geq p_1\,|\,m)}\frac{P(p\geq p_1)}{P(p<p_1)} \tag{7}$$

Let $l=\log_{10}(B)$. Usually, the rule of making the decision is as follows.

If $0<l\leq 0.5$, the evidence against H_0 is poor,
if $0.5<l\leq 1$, it is substantial,
if $1<l\leq 2$, it is strong,
if $l>2$, it is decisive.

Frequently, see e.g. Santer and Duffy (1989), in order to asses the parameters u and z, it is assumed that $u+z=n$ and $E(M)=p_0$ where p_0 is the mean value of the admissible (or expected) level of probability that the audited system works unwell. Hence, on the basis of equation $E(p)=\frac{u}{n}=p_0$ we have: $u=np_0$ and $z=n-u=n(1-p_0)$. This leads to the following: $V(p)=\frac{p_0(1-p_0)}{(n+1)}\approx V(\hat{p})$. Moreover, the parameters u and z can be estimated by means of the well-known empirical Bayes procedure, see e.g. Copas (1972), Griffiths and Krutchokoff (1971) or Walter and Hamdani (1987).

Example 1. The auditor controls 40 accounting documents. He has found that two of them contain errors. It is assumed that the internal control system is good when $p_0=0.03$. The auditor states that the internal control system is wrong when $p\geq 0.08=p_1$ where p_1 is inadmissible probability of finding such documents.

Hence, we have: $m=2, n=40, p_0=0.03, u=1.2, z=38.8$ and

H_0: $p<p_1=0.08$, H_1: $p\geq p_1=0.08$.

The posterior distribution is:

$$h(p\,|\,2)=\frac{\Gamma(80)}{\Gamma(3.2)\Gamma(76.8)}p^{2.2}(1-p)^{75.8}\sim B(3.2,76.8)$$

In this case $E(p|2)=0.04$, $V(p|2)=0.0005$, $\beta_1(p|2)=1.1202$. Using the *R*- function: *pbeta(0.08,3.2,76.8)* we have:

$$P(p<p_1\,|\,2)=0.9463>r=\frac{1}{2}$$

Hence, the auditor should accept the internal control system as well working.

The prior distribution of p is:

$$h(p) = \frac{\Gamma(40)}{\Gamma(1.2)\Gamma(38.8)} p^{0.2}(1-p)^{37.8} \sim B(1.2{,}38.8)$$

In this case $E(p|2)=0.03$, $V(p|2)=0.0007$, $\beta_1(p|2)=1.7874$. Let us note that the variance is larger in the case of the prior distribution than in the posterior distribution. The skewness coefficients satisfy a similar relation. So, the prior distribution of p is more asymmetric than the posterior one which is confirmed by Figures 1 and 2.

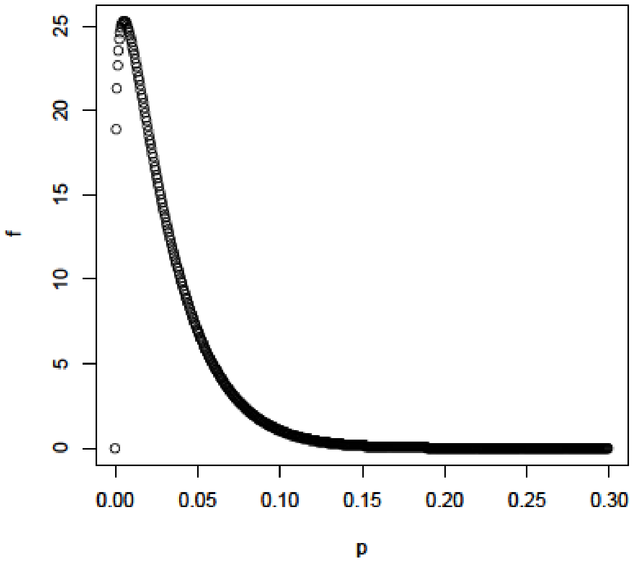

Figure 1. The density of the prior beta distribution B(1.2,38.8).
Source: Own preparation.

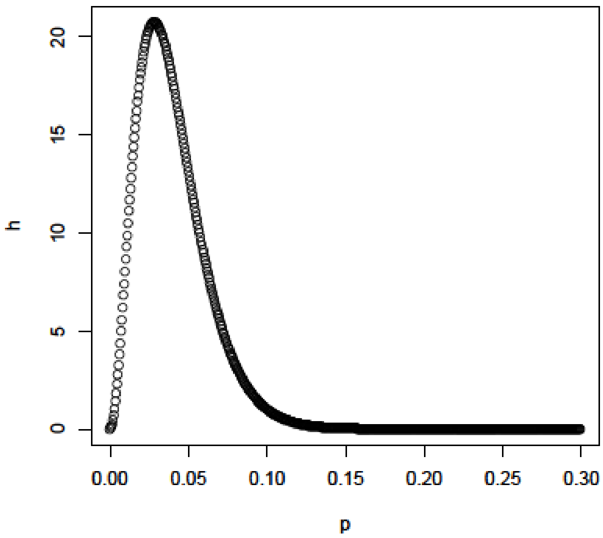

Figure 2. The density of the posterior beta distribution B(3.2,76.8).
Source: Own preparation.

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Using *pbeta(0.08,1.2,38.8)* function call we obtain: $P(p < p_1 = 0.08) = 0.9435$, $P(p \geq 0.08 | 2) = 0.0537$, $P(p \geq 0.08) = 0.0566$.

$B = \frac{1.0031}{0.9490} = 1.0569$, $l=0.0241$.

Concluding, the evidence against good quality of working internal control system (against H_0) is poor because $0<l\leq 0.5$.

2.2.The stratified population

Now the considered system is treated as the non-homogeneous population of subsystems, which are mutually independent. Let us assume that the population is divided into H strata corresponding to homogeneous subsystems. Let p_h be probability that the controlled subsystem is wrong in h-stratum. So, we can write $P(X_h=1)=p_h$, $P(X_h=0)=1-p_h$, $h=1,\ldots,H$.

The following hypotheses are considered:

$$H_0: \quad p = \sum_{h=1}^{H} w_h p_h < p_1, \qquad H_1: \quad p = \sum_{h=1}^{H} w_h p_h \geq p_1 \tag{8}$$

where p_1 is inadmissible level of probability that the audited system works incorrectly. The stated hypotheses are verified on the basis of simple samples drawn from the strata under the traditional frequency approach e.g. by Wendell and Schmee (1996). Here, the following Bayesian model is considered, see Meeden (2003). The simple random sample drawn from h-stratum is denoted by $X_{h,1}, X_{h,2},\ldots, X_{h,n_h}$. The sum $M_h = \sum_{i=1}^{n} X_{h,i}$ has binomial distribution with the following probability function.

$$P(M_h = m_h | p_h) = \binom{n_h}{m_h} p_h^{m_h}(1-p)^{n_h-m_h}, \qquad \text{h=1,…,H.} \tag{9}$$

Let us assume that the beta distribution $B(u_h,z_h)$ is the prior distribution of the probability p_h and its density function is:

$$f_h(p_h) = \frac{\Gamma(u_h + z_h)}{\Gamma(u_h)\Gamma(z_h)} p_h^{u_h-1}(1-p_h)^{z_h-1}, \qquad \text{h=1,…,H.} \tag{10}$$

Under the assumption that random variables $p_1, p_2,\ldots, p_H$ are independent, their joint distribution is of the beta-type with the following density function:

$$f(p_1, p_2,\ldots, p_H) = \prod_{h=1}^{H} f_h(p_h) \tag{11}$$

The posterior distribution of the parameter p_h is:

$$g(p_h \mid m_h) = \frac{\Gamma(n_h + u_h + z_h)}{\Gamma(m_h + u_h)\Gamma(n_h - m_h + z_h)} p_h^{m_h + u_h - 1}(1 - p_h)^{n_h - m_h + z_h - 1} \qquad (12)$$

Simple samples are independently drawn from the strata. So, $m_1, m_2, \ldots, m_H$ are independent and the joint posterior distribution of $p_1, p_2, \ldots, p_H$ has the following density function:

$$g(p_1, p_2, \ldots, p_H \mid m_1, m_2, \ldots, m_H) = \prod_{h=1}^{H} g_h(p_h \mid m_h) \qquad (13)$$

According to the Bayesian rule of testing statistical hypothesis and the expressions (11) and (13) the following probabilities are evaluated:

$$P\left(\sum_{h=1}^{H} w_h p_h < p_1 \middle| m_1, m_2, \ldots, m_H\right) = \int_{\sum_{h=1}^{H} w_h p_h < p_1} g(p_1, p_2, \ldots, p_H \mid m_1, m_2, \ldots, m_H) dp_1 dp_2 \ldots dp_H , \qquad (14)$$

$$P\left(\sum_{h=1}^{H} w_h p_h < p_1\right) = \int_{\sum_{h=1}^{H} w_h p_h < p_1} f(p_1, p_2, \ldots, p_H) dp_1 dp_2 \ldots dp_H . \qquad (15)$$

Those probabilities let us make the decision on accepting or rejecting the hypothesis H_0 on the basis of the rules defined by the expressions (5)-(7). But now there is a problem with calculation of the integral. They can be evaluated on the basis of appropriate numerical methods or through the Monte-Carlo approach.

Example 2. The auditor controls 15, *17* and *18* accounting documents drawn from three strata which fractions are respectively $w_1=0.2$, $w_2=0.3$, $w_3=0.5$. He does not find documents with errors in the samples drawn from the strata. It is assumed that in each stratum the internal control system is good when $p_0=0.03$. The auditor states that the internal control system is wrong when $p \geq 0.07 = p_1$ where p_1 is an inadmissible probability of finding documents with errors.

Hence, we have: $m_1=0$, $n_1=15$, $m_2=0$, $n_2=17$, $m_3=0$, $n_3=18$, $p_0=0.03$, $u_1= p_0 n_1=0.45$, $z_1=n_1-u_1=14.55$, $u_2=p_0 n_2=0.51$, $z_2=n_2-u_2=16.49$, $u_3=p_0 n_3=0.54$, $z_3=n_3-u_3=17.46$. On the basis of the procedure from the Appendix we have:

$$P(0.2p_1 + 0.3p_2 + 0.5p_3 < 0.07 \mid 0,0,0) = \int_{0.2p_1 + 0.3p_2 + 0.5p_3 < 0.07} g(p_1, p_2, p_3 \mid 2,1,3) dp_1 dp_2 \ldots dp_H = 0.9310 ,$$

$$P(0.2p_1 + 0.3p_2 + 0.5p_3 < 0.07) = \int_{0.2p_1 + 0.3p_2 + 0.5p_3 < 0.07} f(p_1, p_2, p_3) dp_1 dp_2 dp_3 = 0.9180 .$$

$B=1.2052$, $\qquad l=0.0811$.

Hence, the evidence against the conclusion that the internal control system works good is poor because $0<l\leq 0.5$.

Example 3. The auditor controls 80*, 40* and *100* accounting documents drawn from three strata which fractions are respectively $w_1=0.2$, $w_2=0.3$, $w_3=0.5$. He has found that there are *1*, *1* and *2* documents with errors in the samples drawn from the *1, 2* and *3* strata, respectively. It is assumed that in each stratum the internal control system is good when $p_0=0.08$. The auditor states that the internal control system is wrong when $p\geq 0.1=p_1$ where p_* is inadmissible probability of finding documents with errors.

The formulated problem can be considered as testing of the following hypotheses

$$H_0: \quad p=\sum_{h=1}^{H} w_h p_h < p_1, \qquad H_1: \quad p=\sum_{h=1}^{H} w_h p_h \geq p_1 \tag{16}$$

After appropriate evaluations involving the computer procedure we have

$$P(0.2p_1+0.3p_2+0.5p_3\geq 0.1|2,1,5)=\int_{0.2p_1+0.3p_2+0.5p_3\geq 0.1} g(p_1,p_2,p_3\,|\,2,1,3)dp_1dp_2...dp_H=0.421<\frac{1}{2}=r,$$

$$P(0.2p_1+0.3p_2+0.5p_3\geq 0.1)=\int_{0.2p_1+0.3p_2+0.5p_3\geq 0.1} f(p_1,p_2,p_3)dp_1dp_2dp_3=0.16\,.$$

$B=3.8174$, $\quad l=0.5818$.

Hence, the evidence against the conclusion that the internal control system does not work properly is substantive because $0.5<l<1$.

3. Conclusions

In conclusion the proposed method is a kind of adaptation of the well-known Bayesian testing statistical hypothesis. The presented Bayesian audit rules seem to be original in the background of statistical inference methods taken into account in auditing, see e.g. Statistical Models and Analysis in Auditing (1989). The Bayesian approach can be seriously considered in the case of small samples and even in the case when all values observed in the sample drawn from binary variable are the same as in Example 2. This approach is based on additional information about the parameters of considered population, which should be known in advance. That information is represented by a prior distribution and its parameters. Usually, the posterior distribution depends on those parameters, which are assessed by means of several reasonable ways. More formally, in practice those parameters are estimated by means of several complex methods like empirical or hierarchical Bayes. Those methods let us improve or modify the proposed testing procedure. In the paper the beta distribution was taken into account as a prior one. Of course it is possible to look for other prior distributions useful in the considered audit problem.

Appendix

The R procedure implementing the evaluation of the expressions (14) and (15) is given below. Observations of H - dimensional prior (posterior) distribution of mutually independent probabilities $[p_1,p_2,\ldots,p_H]$ are evaluated by means of the beta-distribution generator of pseudorandom values. Next, it is checked if the inequality specified by the hypothesis H_0 is true. These operations are replicated a large number of times. Finally, the probabilities given by the equations (14) and (15) are assessed by means of the frequency of the true inequalities.

```
it=10000
p0=0.08
p1=0.1
#number of the strata:
H=3
#the strata fractions:
w=as.matrix(c(0.2,0.3,0.5),H,1);
u=as.matrix(0,H,1); z=as.matrix(0,H,1);
n=as.matrix(c(80,40,100),H,1)
m=as.matrix(c(1,1,2),H,1)
u=p0*n; z=n-u

Bayesfactor=function(p,w,u,z,n,m,it){
# Function implementing the Monte-Carlo integration.
# H0: w1*p1+w2*p2+...+wH*pH<p,
# w - H-element column vector of the strata fractions,
# u=[u_h], z=[z_h], h=1,...,H - vectors of the prior beta distributions B(u_h,z_h),
# n=[n_h], h=1,...,H - vector of the strata sample sizes,
# m=[m_h], h=1,...,H - vector of success in the strata sample,
# it - number of Monte-Carlo iterations,
H=nrow(w)
prior=matrix(0,H,1)
posterior=matrix(0,H,1)
Nprior=0
Nposterior=0
t=1
while (t<=it)
{for (h in 1:H)
 {prior[h]=rbeta(1,u[h],z[h])
  posterior[h]=rbeta(1,m[h]+u[h],m[h]-m[h]+z[h])
 }
 if (t(prior)%*%w<p) Nprior=Nprior+1
 if (t(posterior)%*%w<p) Nposterior=Nposterior+1
 t=t+1
}
Nprior=Nprior/it
Nposterior=Nposterior/it
Bf=Nposterior*(1-Nprior)/((1-Nposterior)*Nprior)
as.matrix(c(Nposterior,Nprior,Bf,log10(Bf)),1,4)
}

Bayesfactor(p1,w,u,z,n,m,it)
```

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