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MATHEMATICS OF UTILITY AND RISK. THREE PAPERS ON STOCHASTIC ECONOMY, FINANCE AND INSURANCE

UTILITY, RISK AND INSURANCE – MAINSTREAMS OF QUANTITATIVE ECONOMICS

(ANNOUNCEMENT AND GENERAL PREFACE TO THE SERIES)

In the short series of three articles we would like to outline the most important ideas, some history and chosen formal facts of the utility theory, risk theory and the so called economics of uncertainty.

The present essay is devoted, primarily, to the description of the key notions, ideas and words of the series – so it is a kind of a dictionary. This is going to be a rather qualitative description of quantitative facts, concerning qualitative categories. We start with a short revision of traditional, natural relations of mathematics and economics, including "personal" relations. Then we are going to discuss the >utility – preferences< relation. Subsequently – we consider the risk, its subjective and objective aspects, measurement and comparison. On this occasion referring to the utility functions, and expected utility is unavoidable. They are the formal tools of assessment or pricing of any economic undertaking, modelled by the so called random elements. In the next two items we will outline the history of the actuarial mathematics and the most important trends of the stochastic financial mathematics of the 20th century. At the end we will refer briefly to the mathematical aspects of the so called hypothesis of rational expectations, which is one of the most attractive and controversial theoretical conceptions in economics at the end of the century.

1. INTRODUCTION: ADAM CONTRA ADAM

For some time now we more often appeal to the classic of the general economic thought Adam Smith than to our native romantic Adam Mickiewicz. Scientific approach "the glasses and eyes" is winning – without question! Signum temporis...

The urgent appearance of numerous qualitatively new phenomena and economic facts has been a natural consequence of the fundamental economic transformation in our country. The collective economic instinct of the nation and the intellectual reflex of individuals (calling it somewhat euphemistically), enabled a pretty big part of society an immediate, favourable adaptation for

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the new conditions. The *sine qua non* condition, however, of the further efficient functioning of the *ad hoc* adopted civilizational devices is their theoretical legitimization. It is necessary to create models functioning as intellectual and legal regulators of the economic element. In the normative sphere the "rules of economic game" have been settled, pointing out the limits of the *laissez–faire* policy, with which the gentlemen of Polish business have not yet been familiar. In the sphere of scientific research the thing is, in the first place, to identify the phenomena enabling formal diagnostics and rationalization of actions.

A fraction of the indispensable devices can be obtained by restoring the old examined standards – applying the obvious modifications. The necessity, however, of restructuring and acceleration of research is urgent.

In the short series of three articles we would like to outline the most important ideas, some history and chosen formal facts of the utility theory, risk theory and the so called economics of uncertainty. The problems have been present in economics "since always". They have been cultivated recently in the economic world with particular intensity both by the theoreticians "in research" and the practitioners "in need". They regard the rationalization of choice and actions (hence the relations with finance, insurance, and intertemporal allocation of investments) as well as the fundamental questions within the borderline of the economic theory, philosophy and logic: the theory of value, righteous or fair price, determinism and indeterminism.

The present essay is devoted, primarily, to the description of the key notions, ideas and words of the series – so it is a kind of a dictionary. *This is going to be a rather qualitative description of quantitative facts, concerning qualitative categories. The paradox is apparent, and its explanation is simultaneous and commonly known: economics is a social science expressed in quantitative categories as well as qualitative ones.* The research apparatus has to be formalized, but "the mathematics can be told" in a less matter-of-fact way, closer to the spirit of the basic economic questions.

We start with a short revision of traditional, natural relations of mathematics and economics, including "personal" relations. The perspective of the end of the second millenium gives the opportunity to see at the some time the Ancient thinkers and the "latest" Nobel prize winners. Then we are going to discuss the >utility – preferences< relation. Subsequently – we consider the risk, its subjective and objective aspects, measurement and comparison. On this occasion referring to the utility functions, and expected utility is unavoidable. They are the formal tools of assessment or pricing of any economic undertaking, modelled by the so called random elements. In the next two items we will outline the history of the actuarial mathematics and the most important trends of the stochastic financial mathematics of the 20th century. At the end we will refer briefly to the mathematical aspects of the so called rational expectations hypothesis, which is one of the most attractive and controversial theoretical conceptions in economics at the end of the century. In references we placed – of course – papers and books which appear in the article. But not only those. Taking into consideration the introductory character of the present work we enclosed some additional positions of unquestionable scientific value, stricly connected with the subject matter.

In the little triptych suggested by the author there are neither economic revelations nor abstract mathematical theorems. There are small formal contributions that are the author's own ideas such as the concept of bilinear functional of utility or the martingale model for the rational expectation hypothesis. They need, however, to work them out and the real confrontation with literature. *The series has a reviewing and, in some sense, promotional character. The subject is not identified well and some items are not mentioned in Polish economic literature at all.* Our (Polish economists' and econometricians') experience and achievements in widely understood econometrics are big, we have results, scattered in time and space, in some fields of mathematical economics (sometimes excellent). We have learnt a lot about insurance and finance for some last years. *One can notice, though, "the grey zone" between the economists' practical consciousness (even theoreticians) and the research area of the "pure" mathematicians. We aim at placing the presented papers in the gap (without any ambition to fill it). We will be fully satisfied if it turns out that we were able to present such contents, to stress such elements, to signal such observations and suggestions that a new quality appeared on the Polish "economic – mathematical market".*

Finally let us notice that we are fully aware of the arbitrary character of the subtitle (Adam contra Adam) but we do hope that Readers will treat this as "licentia poetica".

2. "FIN DE MILLÉNAIRE" IN ECONOMICS AND MATHEMATICS

In the last 25 years the whole economic world achieved "the second cosmic speed". Particularly dynamic development has been noticed in the areas just "resurrected" in our country such as the capital markets, banking and insurance. This also concerns, to a great extent, the theoretical sphere of economics. The principal controversies concerning principles of Keynesists, Neo-Keynesists, monetarists, and neoclassicists of different shades do not abate. There explode new – sometimes apparent – "scientific fireworks". To the "transcendental round table" are asked (or rather summoned!): Plato, Aristotle, Thomas Aquinas, Adam Smith, David Ricardo, Karl Marks, Leon Walras, Karl Menger, Vilfredo Pareto, Alfred Marshall, John M. Keynes. The hosts, in the name of the present *fin de siécle* (*encore!*) could be, for instance: Paul Samuelson, John Hicks, Kenneth J. Arrow, Gérard Debreu and John Nash.

The integration and the attempt at "a great synthesis" of such a polymorphic, multivocal and multicoloured product of civilisation as the economics of the end of the second millenium is a larger-size undertaking than the building of the pyramids. Of course, one can have doubts about the adequacy and reason for such far-fetching analogy, but the former and the latter works share one common feature, which is the necessity of massive, multilateral, purposeful actions for their realization. *The role of the ancient "engineers-architects" has been fulfilled, for 150 years, by economists-mathematicians. Firstly – they give labels to things, secondly – they encourage mutual stimulating of the economic thought trends with the "compatible" with them areas of mathematics, thirdly – they endow economics with measuring instruments.* Functioning as "obstetricians" to the births of model conceptions – they themselves choose a definite theoretical option and thus they participate in creating new schools. Creation of the fundamental categories and laws of economics in the mathematical language enables their precise, scientific analysis, verification and falsification of hypotheses and methodology. In the process of mathematical abstraction one looks for the essence, the gist, and the "principal components" (as Harold Hotelling would say) of economic problems.

Let's look for some time at the history of relations of mathematics and mathematicians with economics. Considerations of the ancient forerunners of economics were of qualitative, ethical-philosophical character. The power of mind of Xenophant, Plato or Aristotle was expressed by mathematical thinking and sophisticated logic of reasoning. In the context par excellence economical Archimedes uttered his famous "eureka" (he was solving, as we know, the problem of defining the gold content in coins). The 13th century logician Duns Scott was engaged in such problems as relations between supply and demand, notions of value and risk. Mikołaj Kopernik (Copernicus) was an expert on monetary policy. A monk Luca Paccioli – a mathematician published in Venice in 1494 a book *Summa di Arithmetica, Geometria, Proportioni et Proportionalitia* regarded by the history of science as the foundation of accounting.

The category of utility in the context of assessment of undertakings of stochastic character was examined by a representative of the famous family of Swiss mathematicians, Daniell Bernoulli. The mathematical aspects of physiocrats' economics are in the programme of subjecting human actions to the universal physical laws of the Universe. François Baudeau wrote: "the science of economics is a real science, as precise as geometry" (Lipiński 1968) whereas economical tables of François Quesnay are typical diagram schemes. A. Smith studied at the Glasgow university mathematics and philosophy of nature. In 1711 he started his lectures there leading the university chairs of logics and the philosophy of morality. No comments... Finally "the father of mathematical economics" Antoine Cournot and his "Recherches sur les principes mathematiques de la theorie des richesses". The title speaks for itself and history added the second meaning to the term "the Cournot point" (the extreme of significant economic function): 1838 – the year of publishing his work. Let us recall the geometrical models and algebraic formulas of Herman H. Gossen, mariginalistic fascinations of William S. Jevons, the Lausanne school with "the engineers of economics" L. Walras and V. Pareto, finally the theoretical culmination of the turn of the centuries – the activities of the mathematician-economist A. Marshall (Fiedor et al. 1979).

The twentieth century. The creators of contemporary mathematical economics were – unanimously – proclaimed John von Neumann, a mathematician-economist, the most brilliant mind of our century and Oscar Morgenstern, an economist-mathematician, Renaissance scientist. (The Polish "Oscar" of modern mathematical economics was Oskar Lange – a high calibre scientist, an active participant of the worldwide dialogue of the makers of the economic theories and methods in the 20th century. He considered as early as in the 1930's the problems of utility quantification – among others. It is important to mention Michał Kalecki – an economist and mathematician well-known in the scientific world.) The consecutive caesure in the mathematical development of economics constitutes year 1944. J. von Neumann and O. Morgenstern published then their famous innovative and at the same time complex monograph "Theory of games and economic behavior". The games mentioned in the title, balance, axiomatization of utility theory – this is "high mathematics" indeed. The apparatus used by Abraham Wald was also refined. This great economist-statistician-mathematician created the theory of statistical decision functions and sequential analysis.

The situation became similar to those we can observe in modern physics: its laws have to be expressed in the language of mathematics. This is not only their language, it is their content! "Fictionization" (belles-lettres) is simply impossible! In economics the phenomenon does not appear so severely – in the background you can always see an economic question. Its indispensable precise statement depends on its transformation into a mathematical structure. The answers are thus, formulated in the language of mathematics. The consecutive "fictionization" requires great care. *The recipe-loke propagation "in simple soldierly words" (a Polish statement) deforms, vulgarizes and loses nuances of the result that can constitute its core.* The Nobel Prize winners in economics were fully aware of this fact. It would be useless to label them according to their acquiring in time new segments of education. Their personalities blow up the conventional classification. It would by childish to stress that J. Nash and Reinhard Selten were first mathematicians and only then economists, and John Harsanyi started with pharmacy. As a rule they gained their education of mathematical and economic orientation at the best American and European universities. So after 1969 "the purse with pearls of today's civilization scattered" about the high class erudites of modern economics and the newest branches of mathematics including topology, functional analysis and stochastic analysis. Let us mention P. A. Samuelson, Jan Tinbergen, K. J. Arrow, J. R. Hicks, Tjalling C. Koopmans, John Tobin, G. Debreu, Robert Lucas. We can only lift our hats to them and to others not mentioned. *Chapeau bas!*

3. UTILITY "VERSUS" PREFERENCES

Utility is one of the key words in economics, sociology and praxeology – together with such categories as value, justice or equilibrium. *Paradoxically: the premise of this theoretical being (some would rather say "the aim of this formal construction"') is objectivization of subjective evaluations and comparisons of goods' values.* The history of economic thought connects the notion of utility, in the first place with the Austrian psychological school i.e. the last decades of the last century. It appears, however, – implicite – in some trends of the ancient so-called "prescientific" economics:

R. D. Collison Black (Collison 1990) writes: "Utility is a term which has a long history in connection with the attempts of philosophers and political economists to explain the phenomenon of value. It has most frequently been given the connotation of "desiredness", or the capacity of a good or service to satisfy a want, of whatever kind. Its use with that meaning can be traced back at least to Gershom Carmichael's 1724 edition of Pufendorf's *De Officio Hominis er civis Iuxta legem Naturalem,* and arguably came down to him through the medieval schoolmen from Aristotle's Politics. Utility in the sense of desiredness is a purely subjective concept, clearly distinct from usefulness or fitness for a purpose – the more normal everyday sense of the word and the first meaning given for it by the *Oxford English Dictionary.*

While most political economists of the 18th and 19th centuries used the term in this subjective sense, the distinction was not always kept clear, most notably in the writings of A. Smith. In a famous passage in the Wealth of Nations A. Smith wrote:

> The word VALUE, it is to be observed has two different meanings, and sometimes expresses the utility of some particular object, and sometimes the power of purchasing other good which the

possession of that object conveys. The one may be called "value in use"; the other, "value in exchange". The things which have the greatest value in use have frequently little or no value in exchange; and, on the contrary, those which have the greatest value in exchange have frequently little or no value in use. Nothing is more useful than water; but it will purchase scarce any thing; scarce any thing can be had in exchange for it. A diamond, on the contrary, has scarce any value in use; but a very great quantity of other goods may frequently be had in exchange for it. (1776, Book I, ch. IV.)"

Let us recall, that from the formal point a utility is a transformation of an economic space with imposed preferences, into a numerical set, preserving these preferences. So we can say that it plays the role of the synthetic representation of the original preferences.

The utility function is not defined *uniquely*: depending on the model and context (deteministic or stochastic) it is determined with the accuracy to the monotonic or solely positive affine transformations. The fact that utility is – in its essence – an ordinal quantity questions the sense of its cardinal interpretation and further consequence i.e. the investigation of the higher derivatives' sign and the role of the integral operator's kernel, used for the risk measurement, which is going to be discussed in the next point. Some investigators, questioning usefulness of utility, set forward an argument of preferences' primordiality in relation to the utility. *Agents' preferences, though subjective – exist objectively but the utility function is created arbitrary.* This is true, though not exactly. (We will discuss this problem in one of the papers.) Other scientists, equally conscious of the mentioned drawbacks of the construction, taking more care of time, do not complain idly but examine mutual dependencies of utility and preference.

From the mathematical point of view we have to do with the problems of existence and the properties of isotonic transformations of topological space with preorders into the linear ordered sets of the real numbers or the finite-dimensional Euclidean spaces with lexicographical order. It is not, however, as seen above, considered in isolation from the economic reality, it is not "harmful abstraction". The formal postulates of preferences' regularity (presented below) and their transforms constitute precise determination of the common sense requirements put to preferences and representing them utility functions.

Let us devote some time to the so called research "path of Cantor-Eilenberg-Debreu": the analysis of isotonic transformations of ordered spaces into the real continuum R. Georg Cantor's statement (1895 – over a hundred years old) claims that, for any set M, completely ordered, without the smallest and the greatest element and containing countable subset M_0, dense – in the sense of order – in M, there exists a preserving order bijection M into R (Beardon, Mehta 1994). In the paper of Samuel Eilenberg (Eilenberg 1941) we have a topological ordered space and the outcome of his work in the language of mathematical economics is: if the consumption set K is a separable and connected space, then the consumer's preferences are representable by some utility function. G. Debreu (1954) proved that if X is a preordered, second countable topological space, then there exists a continuous strictly isotonic function $f: X \to R$. We will present the later results of this trend both positive and negative, as well as their relations with general separation theorems in topological ordered spaces (Monteiro 1987; Braves et al. 1994).

For the last 25 years a different, slightly more general directions of the discussed researches have been parallelly developed. The interest of investigators comprises not only properties of single preferences – assuring the existence of continuous real representation of this relation (roughly speaking: the matter is the type of "compatibility" of the order with topology – continuity or closedness), but a global analysis as well.

In this approach a notion of a joint continuity of utility is introduced and the space of preferences is topologized. The work of Yakar Kannai (1970) has, in this subject, pioneering character. It is devoted to continuity properties of the core of the market, and its "topological aspect" is briefly presented below.

A relation $\succ$ on a space Ω will be called a preference order if it is complete, transitive, and reflexive. If $x \succ y$ we shall say that x is preferred or indifferent to y. If $x \succ y$ and $y \succ x$, we shall say that x is indifferent to y and write $x \approx y$. If $x \succ y$ but not $x \approx y$, we shall say that x is preferred to y and write $x > y$. We shall always assume that there exist vectors x, y such that $x > y$, i.e., we exclude the "trivial" preference. A preference order is called continuous if the set $\{(x, y): x > y\}$ is open in the product space $\Omega \times \Omega$; in other words, if $x_n \to x$, $y_n \to y$, and $x > y$ imply $x_n > y_n$ for all sufficiently large n. It is well known that a continous preference order may be represented by a continuous utility, i.e., there exists a continuous function $u: \Omega \to R$ such that $x > y \Leftrightarrow u(x) > u(y)$, (inequality on the right-hand of the equivalence has to be interpreted as a "ordinary", strong inequality in a set of real numbers).

Denote by Ξ a set of preference orders. We want to introduce a "natural" topology on Ξ. A plausible requirement on a topology on Ξ is that if $x > y$ and if $x_n \to x$, $y_n \to y$ and $>_n \to >$ (in the topology of Ξ), then $x_n >_n y_n$ for all sufficiently large n. In terms of open sets it says that the set $A = \{(x, y, >): x > y\}$ is open in the product space $\Omega \times \Omega \times \Xi$. Therefore, if there is a minimal topology on Ξ which makes the sets A open, then this may be considered a "natural" topology on Ξ. (This notion of natural topology is closely related to the compact – open topology of function spaces.) The author shows that there exists a natural topology on Ξ, characterizes it, and proves that it has a countable basis. Assuming more on $>$, he is able to introduce a metric on Ξ which will induce that topology. A significant step of his construction is arranging the rational balls (i.e., the balls with rational centre and rational radius) in Ω in a sequence. $\{S_i\}$. Next he sets $\Xi_{i,j} = \{> \in \Xi: x > y$ for all $x \in \bar{S}_i$ and $y \in \bar{S}_j\}$ (a superior bar denotes closure). Of course, $\Xi_{i,j}$ can be non-empty only if $\bar{S}_i \cap \bar{S}_j = \emptyset$. Then the following statement holds:

THEOREM: Let Ξ be a set of continuous preference orders. Then the minimal topology on Ξ for which the set $A = \{(x, y, \succ): x > y\}$ is open in $\Omega \times \Omega \times \Xi$ exists and is equal to that topology which has the class $\{E_{i,j}\}$ as a sub-basis.

The use is made of generalizing and refinement of topology generated by the Hausdorf metrics in families of sets, i.e. topologies of closed convergence, open compact topology and, so called, σ and τ topologies (Chichilinski 1980).

The models of situations are examined too, where we are given a measurable space of individuals, each of whom is endowed with a preference relation in their "own" preference field and joint measurability of numerical functions representing their preferences is required for technical purposes – e.g. for integrating (Aumann 1969; Wieczorek 1980).

Close related to the above mentioned models are problems of preference orders in markets with a continuum of traders. One of the latest works in this trend is a paper about the so called selections of Paretian utility functions (Jackson et al. 1994). The authors take into consideration the so called closed linear preorder in the topological space Y (that is the linear order in the set of classes of equivalence relations of indifference generated in the natural way by the initial preorder). Lets us call (Debreu 1964) the continous representation of the preorder of values $\langle 0; 1 \rangle$ a Paretian utility function. The main theorem of the paper is comprised in the following statement:

THEOREM: Let T and X be Polish spaces, R a Borel subset of $T \times X \times X$ such that for each t, $R_t = \{(x, y): (t, x, y) \in R\}$ is a preference order on $B_1 = \{x: (t, x) \in B\}$, where $B = \{(t, x): (t, x, x) \in R\}$ is the field of R. Then there is a Borel measurable function f: $B \to [0, 1]$ such that for all $t \in T, f_t$: $B_t \to [0,1]$ is a Paretian utility or continuous representation of R_t.

The mathematical proof is not elementary: the functional analysis, measure theory and even the foundations of mathematics are exploited (particularly essential role play the theorems connected with continuous selection for multifunctions – this should not surprise, as generalized preferences are correspondences (Vind 1993; Malawski 1995)).

4. RISK AND ITS CERTAINTY EQUIVALENTS

Few regard, at present, that economic behaviour depends on quiet, monotonous work á la the 20th century accountant. Just the opposite: *the economic life is a great, dangerous adventure!* Let us start with some quotations. Mark Machina and Michael Rothshild write (Machina 1990) "The phenomenon of risk (or alternatively, uncertainty or incomplete information) plays a pervasive role in economic life. Without it the financial and capital markets would consist of the exchange of a single instrument in each period, the communication industry would cease to exist and the profession of investment banking would reduce to that of accounting". Then David Begg, Stanley Fisher and Rudiger Dornbuch noticed (Begg 1993): *"There are only two certain things in the world: that we will die and that we have to pay taxes"*. The world is full of risk, and attitudes of people towards it are differentiated: begining with the most popular risk aversion, through risk neutrality to risk attraction and gambling.

The most basic form problems connected with uncertainty and risk can be presented in four groups of issues:

a) issues connected with definitions (including the fundamental though conventional differentiation of "the stochastically determined risk" from "the strategically indeterminate uncertainty" – traditionelly derived from Frank Knight (Machina 1990));

b) methods of measurement and comparing risk;

c) quantitative characteristics of attitude to risk;

d) mutual relations between the utility functions of economic agents and measures of "their" risk (as well as measures of their attitude to risk).

The subject matter will be discussed wider in the paper "Evaluation and Ordering of Risks". Let us now draw your attention to some chosen theoretical trends and present key words. We want to stress that the aims of measuring and comparing risk are indispensably connected with the principal question of defining it. Nearly general consensus was reached in this matter: *the risk is, formally, identified with its "bearer" or "generator" i.e. a random element in a sufficiently rich mathematical space* (or probability measure – distribution of the above mentioned variable on this space). This is also the theoretical idealization of the so-called lotterie – in von Neumann-Morgenstern, Luce--Riffa meaning. A lottery is any decisive situation in stochastic conditions. Such a formalization is applied in the analysis of financial prospects, classification and tariffication of the so-called actuarial risks are, in general, in any context connected with investments.

Thus it is necessary to measure and compare random elements or/and their distribution. Rendering this mathematically: *this is the problem of the stochastic orders*. On event of the distribution of stresses and purposeful comparative aspects, different kinds of stochastic orders are considered. The bibliography is immense. New propositions appear, adequate to new situations in "the operational spheres of the subject matter", as the new phenomena in actuarial mathematics and attempts of the global comparisons of the whole risk processes.

One of the most intelligible and general methods of determining orders in families of probability distributions on a given space has been provided by the idea of *stochastic dominance*. Let us describe, briefly, the general features of this methodology (Mosler et al 1991, p. 262-284). Let $\mathscr{A}$ be a family of sets of field $\mathscr{B}$ of the subsets of space X. We define, on the set P of probabilistic measures on $(X, \mathscr{B})$, the relation of the partial preorder $\succ_{\mathscr{A}}$: for given measures P and $Q \in \boldsymbol{P}$

$$P \prec_{\mathscr{A}} Q \Leftrightarrow P(A) \leqslant Q(A) \quad \text{for each} \quad A \in \mathscr{A}. \tag{1}$$

We obtained the so-called stochastic dominance generated by the family $\mathscr{A}$. Measures of sets can be identified with integrals of their indicators (with respect to these measures). Therefore the right side of the relation can be described in this form:

$$\int_X I_A(x) P(dx) \leqslant \int_X I_A(x) Q(dx), \qquad A \in \mathscr{A}. \tag{2}$$

Out of the integral's linearity results preservation of inequality, if the indicators under integrals are replaced by their positive linear combinations. Then the Lebesgue theorem on the monotonic convergence makes possible generalization of underintegral function – these can be any integrable limits of increasing sequences of non-negative simple functions, $\mathscr{A}$-measurable. Thus we can state that there is:

$$\int_X u(x) P(dx) \leqslant \int_X u(x) Q(dx), \tag{3}$$

for all functions u of a closed, convex cone $\mathscr{F}_{\mathscr{A}}$, generated by the characteristic function of sets belonging to the family A. We can, duly, treat a fixed cone $\mathscr{F}$ of integrable functions as a generator of preorder. From the scientific point of view it is attractive to characterize cones $\mathscr{F}_{\mathscr{A}}$ for given families $\mathscr{A}$.

So the instrument of measurements and comparing of risks is an integral of a given function with respect to the risk – the probabilistic measure. Attention! Right now we enter a platform that links the risk theory to the utility theory. This platform is called the expected utility. If, for instance, in a consumer theory we take an assumption that an individual person is able to create a subjective ranking on a set of probability measures (with which they can, potentially, have something to do) and their preferences fulfil the "rational" conditions of regularity, then they have the real representation (vide: the previous paragraph of the paper). This is the so *called real valued preference function or utility functional over the set of the probability measures* (Machina 1990). However, such "classic" sets of assumptions imply nothing about the functional form of the utility functional V (as in nonstochastic case). For reasons of both the normative appeal and the analytic convenience, economists typically assume that V is a linear functional of the underlying measure P (or its distribution function) and hence takes the form:

$$V(P) = \int u(x)\, P(\mathrm{dx}) \qquad (4)$$

for some function "u" over wealth levels x, where "u" is referred to as the individual von Neumann-Morgenstern utility function. (Relation (4) reflects the well-known idea that, roughly speaking, "expectation = integration").

It is worth to pay attention to the two aspects of the above construction. Firstly: the utility functional and the utility function coincide on probability measures with masses concentrated in single points – deterministic ones. We obtained, thus, a general *method of extension of orders on a given space to a set of measures on this space* (Wieczorek 1978). Secondly: *we built a scalar index characterizing risk synthetically*. As it is known, *it is called a certainty equivalent of risk* (for a subject with a given utility function u). A discussion of the second aspect is the subject of the so called expected utility hypothesis. The hypothesis has a long and rich history. It goes back to Nicolas and Daniel Bernuolli, Gabriel Cramer, and Pierre Simon Laplace. It is connected with the so called Petersburg paradox (Samuelson 1977) which... is not a paradox at all. It was an empiric suggestion of giving up the linear functions of the utility of wealthy, sometimes leading to irrational speculations. It was used productively by D. Bernoulli – a precursor of the certainty equivalent of game – using the logarithmic utility function.

The expected utility hypothesis itself is not a hypothesis either! This is, in fact, a series of theorems, expressing the existence of integral representation for suitable extention of the initial utility function. It is obvious that the introduction ordinal and topological (or measurable) structures for economic spaces is necessary. Assumptions of consistency are indispensible when the space is extended to the space of probabilistic measures defined on it – the so called conditions of J. von Neumann: continuity axiom, independence axiom, unequal probability axiom (Findley et al 1978). In reward, we obtain the consequent extension of the classic utility function to a functional defined on the space of distributions on the space of goods – with the historic name "the Bernoulli utility index".

The so called "research path of Ramsey – von Neumann and Morgenstern – Herstein and Milnor" (Herstein et al 1953) leads us to a stylish synthesis of the obtained results. This is the so called Grandmont theorem of the expected utility.

THEOREM (Grandmont 1972, Miyake 1990). Let X be a separable metric space, $P(X)$ – a space of probability measures on X, endowed with weak convergence topology and relation of the continuous preferences of von Neumann-Morgenstern (von Neumann 1944, p. 26-29, 617-628). Then:

(A) There exists a continuous isotonic real function u: $P(X) \to R$, which restriction to X (i.e. some function u_X: $X \to R$) is continuous and bounded, such that:

$$u(p) = \int_X u_X(x)\,dp(x) \qquad \text{for all } p \in P(X) \tag{5}$$

(the above statement may be regarded as the modern version of the mentioned earlier famous "Expected Utility Hypothesis").

(B) The function u is determined uniqually up to the positive affine transformations: every function v satisfying the thesis (5) is of the shape:

$$v(p) = \alpha \cdot u(p) + \beta, \tag{6}$$

where $\alpha > 0$, $\beta \in R$ – real constants, $p \in P(X)$.

If c denotes the certainty equivalent of a "risky situation" X, then the *arithmetic difference between mathematical expectation of* X (in the real random variable case), *say* μ, *and c is called the decision maker's risk premium.* Denoting the risk premium by Π, the notational definition is:

$$\Pi = \mu - c. \tag{7}$$

Decision makers whose utility functions produce positive, zero, and negative risk premiums are said to exhibit risk aversion, risk neutrality and risk attraction, respectively. These properties in turn are associated with "behaviour" of the marginal utility – exhibited, in principle, by a sign of the second derivative of function u. Perhaps it is worth to make some remarks on the role of a certain equivalent c of X and a risk premium Π. Firstly: a decision maker would be indefferent between having c (for instance – in monetary units) with certainty and facing risk X (lottery, financial prospect). The interpretation of the role of the risk premium is twofold: it represents the extra amount that the decision maker must have in order to bear risk. On the other hand, it represents the extra amount that the decision maker is prepared to forego in order to avoid risk.

"The Bishop" of mathematical economics K. J. Arrow, writes in his "*Essays in the Theory of Risk-Bearing...* (Arrow 1971): the shifting of risks, the very essence of insurance, occurs in many forms in the economic systems..." He writes: "...the failures of the market to achieve adequate risk shifting lead to compensatory alterations in social institutions, licensing, bankruptcy and limited liability, and large business organizations. But all of these institutions are steps away from the free working of the price system, which, with the defects that have been noted, has also many virtues which do not need to be expanded on here. Especially, we expect all these institutions to decrease the flexibility and responsiveness of the system to change and innovation".

Fortunately, human behaviour adapts to uncertainty and risk in a variety of ways. The insurance, future markets and the use of stock markets are the most important institutions that facilitate adaptation to the risk aversion. The next paragraphs are devoted to report some facts from the history of insurance and the main problems of financial mathematics.

5. SOME HISTORICAL REMARKS ON ACTUARIAL MATHEMATICS AND RISK PROCESSES[1]

Homo Sapiens from the very beginning had to battle against difficulties which he euphemistically called random events, as he could not solve them. He started to think. He devised repressive actions of suppressing and preventive character. He saved, divided "big risk" into several "little risks". Then he changed into Homo Oeconomicus – that is a foreseeing manager. But the calamities were not weaker any more. They changed slightly but remained the same: lions tear camels to pieces, and merchants, storms and pirates attempt sailors lives, and lightnings set fires...

The Indians, Babylonians and Phoenicians unite in order to bear risk in a group. They are also familiar with charging payments for taking risk. The pious Israelites collectively gather money to endow poor virgins. The Romans create military and funeral savings banks. In the late Middle Ages come into existence German "Feuergilden" (or "Feuerkassen") and Italian "foenus nauticum" (a marine loan known in the ancient times). The notion "assecurantum" was found and the institution of a broker, i.e. an intermediary of sides and the arranger of a transaction. Lorenzo Tonti of Naples thinks out state pensions for the state treasury of France – the famous tontines. The English have mutual assistance societies or "Friendly Societies".

The main idea and aim of all these institutions and forms of a group, mutual assecuration was the so called "equalization of risks". These were economic devices covering future material needs brought about in particular units (of a society) by random events, occuring in some regular manner. Homo oeconomicus is also a gambler. He plays not only for biological or economic survival but he is eager to take risk to gain profit. Such a risk is taken by the participants of the insurance transactions in Hanseatic League towns.

The first stock companies in the marine insurance are profit oriented (1602 the Dutch-Indian Society was set up). *The necessary condition of success in the game with Nature is the ability of quantification of random insurance events' dynamics. This applies, in general, to the possibility of functioning and is the*

[1] The contents this paragraph was taken – mainly – from the paper (Rybicki 1995).

minimum condition of the "physical realization" of any insurance institution. As early as tontines operations required analysis of demographic processes as the basis for financial calculations. "Homo oeconomicus" *nolens volens* undergoes some mutation, "homo mathematicus" appears on the stage (still oeconomicus!). *He aspires to recognize in seemingly chaotic caprices of fate "the logics of events". To achieve this, one has to create the necessary instruments and models constituting the actuarial knowledge and insurance mathematics.* This discipline can be generally defined as the art of modelling and measuring of random processes. Their material consequences should be neutralized by the parallel creation of the proper deterministic processes (or at least steered) – financial streams. At the very beginning of the insurance development we can differentiate two main branches of the actuarial theory. One of them is created by theories and methods of life insurances, for which the forerunner were the tontines, Jan de Wit research and finally, the first in history death – rate tables arranged by the famous English astronomer Edmund Halley in 1693, on the basis of death records from Wrocław. The date is symbolically accepted as the moment of the life insurance's birth and E. Halley is its father.

The second branch of the actuarial theory reflects the intellectual duel of the economist-mathematician with the dynamic random process. As the ancestor of this risk theory we should count D. Bernoulli. Later A. S. Laplace was engaged – *ex definitione* – in probability of games, risk, the problem of gambler ruin. The proper creator of the dynamic theory of risk is a Swedish scientist P. Lundberg (1903) who at the beginning of this century proposed stochastic models to observe insurance accidents in time. He introduced the notion of the risk process and the so-called individual and collective risk, which is functioning in the theory till the present time. The concept of the random point process instituted (as the price process of Bachalier's in financial mathematics and the Erlang's model in queuing theory) a contribution for creating the theory of stochastic processes. If we say that Ph. Lundberg is a father of the contemporary risk theory, then equally bright godfathers are Harald Cramer nad Bruno de Finetti.

Ph. Lundberg treated the stream of claims for compensations as "purely random process of points, appearing in time, with constant intensity λ". The mathematical consequence of the model postulate was, the so called ordinary stochastic point process

$$N = \{N_t, t \geqslant 0\}, \tag{8}$$

with the stationary and independent increments. Exactly the same Poisson process that simultaneously happened to find out engineer Erlang. Let's recall: the probability of exact occurance of "k" claims for compensation in any interval of length $s > 0$, is:

$$P(N_{t+s}-N_t = k) = \frac{(\lambda s)^k}{k!} e^{-\lambda s}, \quad k = 0, 1, \ldots \tag{9}$$

Let random variable $Y_j (j = 0, 1, \ldots)$ denotes volume of compensation connected with j-th insurance claim. Then the dynamics of changes of the accumulated compensation sum will be described by the compound process:

$$X = \{X_t, t \geqslant 0\}, \quad X_t = \sum_{k=0}^{N_t} Y_k. \tag{10}$$

The last element of Lundberg's model is (random) function, describing fluctuations of the insurer's capital in time:

$$S = \{S_t, t \geqslant 0\}, \quad S_t = X_0 + ct - Y_t, \tag{11}$$

where X_0 denotes the initial capital of the society, and the positive constant c is the intensity of income growth, N – process of the of the consecutive calls (portfolio claims), Y – simple claims process, X-aggregated claims process, S-surplus process. Processes X, Y, S are generally called the risk processes.

John C. Hickman (Hickman et al 1986) in his "Introduction to Actuarial Mathematics" distinguishes the following basic parts of actuarial mathematics (which he determines as "a collection of mathematical ideas that has been found useful in designing and managing financial security systems").

A – Long term insurance, which, in turn, is divided on:
 (1) Individual life insurance,
 (2) Private pensions,
 (3) Social insurance.

B – Short-term insurance, the main blocks of it are:
 (1) Risk models and loss distributions,
 (2) Credibility theory.

Paul Embrechts and Claudia Klüppelberg write: "The amalgamation of relevant theory from diverse fields has now resulted in the emergence of a full-bodied branch of science called 'Insurance Mathematics'. A glance at the variety of topics included in this theory reveals such names as:

- Risk theory,
- Life insurance mathematics,
- Premium rating,
- Credibility theory,
- Pension funding,
- Solvency studies,
- Population theory,
- IBNR (Incurred But Not Reported) modelling,

– Reserving,
– Insurance and mathematical theory of finance,
– Reinsurance,
– Survival modelling and loss distributions." (Embrechts and Klüppelberg 1993).

In a contemporary actuarial mathematics more and more complicated stochastic processes have begun to be exploited: compound Poisson processes, mixed Poisson processes; double stochastic Poisson processes; general point processes, as well martingales. We are going to report the present status of this theory in one of the annunced papers.

6. INFORMATION ON THE STOCHASTIC ANALYSIS OF FINANCE

The times when the application of mathematics in finance resolved itself into a compound interest and derivative problems: discounting, regulations of repaying a credit, and even the analysis of bonds' value and the earmarked funds, made history. Remained the universal invariants – exponential laws of the growth. The increase of the scale of capital's turnover, globalization of phenomena and financial operations, revolutionary acceleration of communication between the world's economic centres resulted in the quantitative and qualitative development of financial sphere. The most spectacular acceleration can be observed in the last 25 years, but the phenomena have already lasted for the whole century. The global commodity and monetary exchanges have great turnover. There arise new possibilities of making money and, parallelly, new threats to wealth's loss.

It is followed by new intellectual speculations, new operations both "active" and "hedging". Let us stress the role of risk and uncertainty whose quantification and decoding has become the main subjects of search for financial practicians and theorists. Equally important means of confrontation with the financial risk have become new instruments of the so called financial engineering: the whole range of futures, forward and derivatives. Their destination is, to some extent, two-way: primary-assuring, secondary-speculative. The dynamics of their development is well illustrated by the following statement: on the 26th of April 1973, when the Chicago Board Option Exchange (CBOE) was opened, 911 contracts for call options of the 16 kinds of shares were realized. A year later 20,000 contracts, 3 years later 100,000, and in 1987 about 700,000 contracts a day (Širiaiew 1994). Approximately 300 years ago Edward Lloyd, the owner of a coffehouse in London realised the need for insurance covering the transport risk connected with the shipping industry.

Today the Lloyd's of London has a premium income of more than 20 million each workday. The contemporary world industry which offers financial services takes over the risk due to risk exposure is now enormous (Embrechts and Klüppelberg 1993).

Naturally, the classic problems of qualifying the value of money in time, investment pricing (in general – multiparametrical), and generally, the problems of utility of wealth expressed in monetary units, have not been deprived of their interest. However the more complicated reality extorted refinement of models and analyses – in spite of necessary simplifications. Simplifying to the limit of vulgarization, we can say that the deterministic rates of growth were substituted by the stochastic rates and the only "traces of old ones" are, at present, the expected return and their standard deviations.

Clifford W. Smith Jr. (Smith 1986) distinguishes several so called "Major Building Blocks of Finance". The efficient market theory is ultimately just a statement of the pricing implications of competition in speculative markets. It hypothesizes that economic profits are impossible from trading on available information. The portfolio theory examines the optimum security selection procedures for an investor's entire portfolio of securities. The asset pricing theory addresses the determinations of assets' prices under uncertainty. William Sharpe and John Lintner (Smith 1986) solve for equilibrium security prices, given that investor demands for securities are implied by the Harry Markowitz mean-variance model (Markowitz 1952). The option pricing theory involves the analysis of the determinants of the prices of contingent claims, the simplest of which are called options. While the capital asset pricing models explain equilibrium expected returns, and thus relates today's asset price and the asset's expected future price, the option pricing model links today's value of a contingent asset with today's value of the underlying asset. Fischer Black and Myron Scholes (Black et al. 1973) derived the solution to the valuations problem for calls. The last branch is, the so called agency theory, providing a framework for the analysis of contractual relations.

It is possible to introduce conventionally, the other division: into a static (or quasistatic) analysis and dynamic analysis. The portfolio theory and asset pricing theory are closer to the former analysis. There appear tasks of comparing random elements and measures, the utility questions, stochastic dominance, effective sets. These problems have been thoroughly discussed in Polish financial literature of the last few years (Jajuga et al. 1996; Smaga 1995). In one of the works of the series I have the intention to remind only the trends of theoretical research on the criteria of effectiveness and the shattering criticism of Karl Borch (Borch 1969), the methodology of mean-variance of H. Markowitz and J. Tobin, which was the beginning of the end of its mechanical application. We will give more attention to the stochastic dynamics of financial phenomena (the product of this trend has been among others the mentioned-above paper of Black and Scholes). Here "career" was made by stochastic processes – natural mathematical abstracts of repeatable random games, but in the last decades – particularly popular have become the so called martingales.

Let us start with the very beginning of the century. On March 29,1900, L. Bachelier presented his doctor's dissertation at Paris Academy (dedicated to his scientific patron – Henri Poincaré entitled *Théorie de la speculation* (Bachelier 1900) – historically the first attempt of a mathematical description of the capital market processes. This was a pioneering work of the stochastic processes theory: L. Bachelier deduced "the Wiener's process mechanics" 25 years before "the official" mathematics – including the great namesake of the process! The work was also precursory in financial mathematics – anticipating modern stochastic models in the field by more than half a century.

In 1965 P. Samuelson (1965) on the initiative of L. Savage paid attention to L. Bachelier's model and modified it. Instead of the original representation of the price process by the formula:

$$S_t = S_0 + \mu t + \sigma W_t \tag{11}$$

(the Brownian movement of the initial state S_0) he considered the so called geometrical Brownian movement (called by him "economic" as well) given by formula:

$$S_t = S_0\, e^{\mu t}\, e^{\sigma W_t - \frac{\sigma^2 t}{2}}, \qquad S_0 > 0, \tag{12}$$

where $W = (W_t, t \geqslant 0)$ is a standard Brownian movement (Širiaiev 1994).

A discrete analogon of the process is the so called geometric random walk $S = (S_m, m \in N)$ expressing the price of a share at the moment n by the formula:

$$S_n = S_0 (1 + \rho_1)(1 + \rho_2) \ldots (1 + \rho_n), \tag{13}$$

where $\rho_1, \rho_2, \ldots$ – a sequence of independent and identically distributed (i.i.d.) binominal random variables taking values a and $b\,(-1 < a < b)$ with probabilities p and $1-p$, respectively. Out of this follows immediately that the differences of this process are determined by the formula:

$$\Delta S_n = p_n^{S_n} \tag{14}$$

(this is the so called CRR model – Cox, Ross, Rubinstein model (Širiaiev 1994)).

Further research in this direction leads to the model:

$$S_t = \frac{S_0}{B_0} B_t e^{\sigma W_t - \frac{\sigma^2 t}{2}} \tag{15}$$

in which appears also the second process B_t – of bonds prices (much more "accurate"). From the formal point of view the last identity shows the so called multiplicative Doob-Meyer decomposition of a process S_t, where $B = (B_t, t \geqslant 0)$ is the so called predictable, increasing process and the second factor:

$$M = \left(\left(\frac{S_0}{B_0}\right)e^{\sigma W_t - \frac{\sigma^2 t}{2}},\ t \geqslant 0\right) \tag{16}$$

is a martingale (Širiaiev 1994). Let us remind that the idea of a martingale $M = (M_t, t \in T)$ with respect to a given sequence of histories (or informative sets) $F = (F_t, t \in T)$, is expressed in postulates of the existence of finite mathematical expectations of variables M_t of this process and "conditional stability" with respect to an increasing informative stream F:

$$E(M_{t+s}|F_t) = M_t \text{ (almost sure)}, \tag{17}$$

(The French word "martingale" denotes an element of harness: an arrangement of leather straps, used for keeping the horse's head at the constant height while horse-riding. J. Ville cained the word into mathematics in 1939 (Ville 1939), natricing the analogy of "the suppressed gallop" with the mechanics of a definite class of random processes.)

Martingales generalize the i.i.d., sequences of random variables, random walks, independent increment processes (among others: the Wiener process, gamma process, some point processes). At present they have been well investigated theoretically and they undergo statistical treatment. The most important are, however, their economic and game-theoretical connotations, with the so called theory of fair games and the absolutely fair random sequences (Feller 1969). Every martingale is – in some sense – a sequence of partial (accumulated) sums of an absolutely fair sequence. Hence the model relations and generalizations of the earlier price movements descriptions: "the random walk hypothesis" and "the fair game hypothesis" (Fama 1970). They also enable elegant formalization of gradation of effectiveness of capital markets (Arrow et al. 1981).

Eugene Fama (1970) distinguishes between three different types of information: information contained in past prices of the securities in questions, $F^{(1)}$; information contained not only in the past prices but also in all past events that have been publicly reported, $F^{(2)}$; and, the information contained in all past events, $F^{(3)}$. $F^{(1)} \subset F^{(2)} \subset F^{(3)}$ and the question can be formally stated as: after allowance for a "normal" rate of return, is the sequence of prices a martingale with respect to $F^{(1)}$, $F^{(2)}$ or $F^{(3)}$ ("weak, semi-strong, strong efficiency respectively"). It should be pointed out that in considering richer, than formal record of a movement of examined process, sequences of histories (including also the exogenous — with respect to the process — information) provides a partial solution of a riddle — paradox of the so-called rational expectation hypothesis (Muth 1961; Lucas 1972).

The following example is an attempt of the simpliest martingale model generalizing the classic pattern of the financial mathematics. It is the so called force of interest martingale (Gerber 1979). Let Y_0, Y_1, Y_2 be a sequence of random variables such that, for each k, Y_k is a F_k-measurable, where $F = (F_k, k = 0, 1, 2 \ldots)$ is an increasing family of sub σ-fields in a given probability space $(\Omega, \mathscr{P}, P)$ (F constitutes a sequence of histories of some financial phenomenon). Let us assume that $\operatorname{sgn} E(Y_{k+1}|F_k) \cdot Y_k = 1$ and let $(\delta_k\colon k = 0, 1 \ldots)$ be a sequence of F_k-measurable functions defined as the solution of the equations:

$$E(Y_{k+1}|F_k) = e^{\delta k} \cdot Y_k,\ S_0 = Y_0 \tag{18}$$

and define:

$$S_0 = \exp\Big(-\sum_{c=0}^{k-1} \delta_i\Big) \cdot Y_k. \tag{19}$$

Then (S_k) is a martingale with respect to (F_k). Note that δ_k can be interpreted as a force of interest that operates between time k and $k+1$ (given the history F_k). In this sense S_k is the present value of Y_k. Of special interest is the case of a constant force of interest i.e. where δ_k is an independent of k and F_k.

7. CONCLUSIONS

In this short part we proceed to formulate some natural, almost tautological, corollaries following from previous considerations.

First of all: the proper set of tools for solving the majority of economic problems constitutes mathematical apparatus. Validity of this statement posesses the features of the objective, everlasting, obvious truth.

Secondly: the subject matter of inter-relationships between the categories of value and price (evaluation) of goods, undertakings, and labour has been still of primary interest both for economists and philosophers — beginning with the triade of Thomas Aquinas "bonitas rei — valour intrinsecus — iustum pretium" till the so called problem of transformation of D. Ricardo and K. Marks. A wide class of general, efficient and logically coherent models for quantifications and comparisons in the above mentioned area is provided by a mathematical theory of ordered sets and their isotonic transformations. In contemporary mathematical economics they function as the preferences theory and generalized utility functions — however it is not the only philosophy and approach to these problems.

Thirdly: the demand for refinement and constant efforts to achieve maximum adequacy of the models requires taking into consideration elements of randomness and uncertainty of economic world. This leads to a conclusion, that the mainstream of modern quantitative economics constitutes the stochastic economy. In this case, we do agree that the belief can be controversial. On the other hand, however, it is common knowledge that essence of existence and functioning of two immense branches of the whole today's economic activity – finance and insurance – is placed in stochastic character of these phenomena. It is worth mentioning, at this point, that financial and insurance economics have recently become a sort of "laboratory" (or "microcosmos") for the entire economy.

In this paper we have basicly attempted to motivate our interest in mathematical formalization of notions and mechanisms of stochastic economy in general. The next two parts of the prepared series will be devoted to a more detailed questions and models:

a) stochastic dynamics of insurance and finance,

b) evoluation and ordering of risk in connection with generalized expected utility theory.

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