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Phase circular hologram as a laser beam splitter*

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Theoretical study of the general order Hermite–Gaussian beam transformation by a phase circular zone hologram showed its applicability as a multiple laser beam splitter, provided the incidence is off-axial. Each of the splitted components is of the same mode-order and orientation as the incident beam, but it is described by different complex parameters. Their waist locations and magnifications are dictated by the positions of the manifold foci of the zone hologram, and for a given diffracting order, satisfy the Self relations, typical for the beam transformation by ordinary lens. From the theoretical results the Kogelnik ABCD rules and "ray" transfer matrices for the CH are defined.

1. Introduction

Realized as an interference pattern of a spherical and plane wave on a bleached phase sensitive material [1], the phase circular hologram (phase CH) represents a phase diffracting device with sinusoidal modulation of either thickness or refractive index or both, according to the fringe pattern. It is a phase analogue to the circular zone hologram [2], [3], and belongs to the wider class of circular zone gratings, which include sinusoidal, square, quasitrapezoid or binary profiles of trasparency and phase modulations of the registered interference fringes.

The zone plates of the transparent type are known as multifocal optical devices, suitable as lens substitutes for experiments in the UV region [4], [5], X-ray astronomy [6], tomography and nuclear medicine [7], laser fusion experiments [8], etc. The advantage of their phase analogues is in much higher diffracting efficiency which may be achieved [9], [10].

The effects of spherical and plane wave illumination of the circular zone plates were studied by many authors [11]–[14]. As far as the laser beam illumination is concerned, only the effects of the transparent zone plates on the fundamental beam modes were studied [15]–[17].

The purpose of this work is to investigate one of the phase versions of zone plates. Since we are interested in the complete picture of the beam transformation that can occur, we have chosen the incident beam to be a general order simple astigmatic Hermite–Gaussian laser beam, so that we can get an information not only on the complex beam parameters changes, but also mode structure transformations.

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The analysis is based on the results obtained from the solution of the Fresnel–Kirchhoff diffraction integral for an off-axial normal beam incidence on a phase CH, since the beam splitting is proportional to the amount of the off-axial displacement.

2. Phase circular wave hologram

Realized by bleaching the transparency of registered interference fringes of a circular and plane wave, the phase circular hologram represents a phase layer with: i) constant refractive index and a thickness relief, ii) constant thickness and spatial modulation of the refractive index, iii) coexistence of both modulations of the thickness and refractive index. It gives a phase retardation to the transmitted light defined by the function

$$T(x_1', x_2') = e^{ik\left[\alpha+\beta\cos\frac{\pi}{r_0^2}(x_1'^2+x_2'^2)+\gamma\cos\frac{2\pi}{r_0^2}(x_1'^2+x_2'^2)\right]}$$
$$= e^{ik\alpha}\sum_{l=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} i^{l+m} J_l(k\beta)J_m(k\gamma)\prod_{j=1}^{2}\exp\left[i\frac{k}{2}(l+2m)\frac{\lambda}{r_0^2}x_j'^2\right], \quad (1)$$

x_j' ($j = 1, 2$) are rectangular coordinates in the plane of the phase hologram, $k = 2\pi/\lambda$ is the propagation constant, α, β and γ are phase layer coefficients depending on the thickness d and refractive index n of the layer in the following way:

$$\text{i)}\ \begin{cases}\alpha = (n-1)d_0\\ \beta = (n-1)d'\\ \gamma = 0\end{cases} \quad \text{if} \quad n = \text{const} \quad \begin{array}{l} d = d_0 + d'\cos\frac{\pi}{r_0^2}(x_1'^2+x_2'^2),\\ d_0 - d' > 0\end{array} \quad (2)$$

$$\text{ii)}\ \begin{cases}\alpha = (n_0-1)d\\ \beta = n'd\\ \gamma = 0\end{cases} \quad \text{if} \quad d = \text{const} \quad \begin{array}{l} n = n_0 + n'\cos\frac{\pi}{r_0^2}(x_1'^2+x_2'^2),\\ n_0 - n' > 0\end{array} \quad (3)$$

$$\text{iii)}\ \begin{cases}\alpha = (n_0-1)d_0 + \frac{n'd'}{2}\\ \beta = n'd_0 + n_0d' - d'\\ \gamma = \frac{n'd'}{2}\end{cases} \quad \text{if} \quad \begin{array}{l} d = d_0 + d'\cos\frac{\pi}{r_0^2}(x_1'^2+x_2'^2),\\ n = n_0 + n'\cos\frac{\pi}{r_0^2}(x_1'^2+x_2'^2),\end{array} \quad (4)$$

$J_l(k\beta)$ and $J_m(k\gamma)$ are integer order Bessel functions and, r_0 is the radius of the central circle bounded by the first extremum of the phase modulation. When the phase CH is of the type i) or ii) ($\gamma = 0$ due to $n' = 0$ or $d' = 0$) only the $m = 0$ term in expression (1) exists, and the transmission function is given by

$$T(x_1', x_2') = e^{ik\alpha}\sum_l i^l J_l(k\beta) \prod_{j=1}^{2} \exp\left[i\frac{k}{2} l \frac{\lambda}{r_0^2} x_j'^2\right]. \tag{5}$$

We shall carry out our study considering the general iii)-type phase hologram, since the results can be easily specified by putting $m = 0$.

By introducing a new summation index

$$\mu = l + 2m,$$

expression (1) can be written as

$$T(x_1', x_2') = e^{ik\alpha}\sum_m \sum_\mu i^{\mu-m} J_{\mu-2m}(k\beta) J_m(k\gamma) \prod_{j=1}^{2} \exp\left[i\frac{k}{2}\mu\frac{\lambda}{r_0^2} x_j'^2\right]. \tag{6}$$

The phase CH represents a diffracting device whose effects on the laser beams are the subject of our investigation.

3. Characteristics of the input beam

The phase CH is illuminated by a laser beam whose incidence is normal and off-axial with respect to the hologram axis, chosen as z-axis of the spatial coordinate system.

The output modes of radiation of the conventional lasers are described by the general order Hermite–Gaussian beams. With propagation axis

$$x_j' = \xi_j = \text{const}, \quad j = 1, 2, \tag{7}$$

the beam is represented by [18]

$$\begin{aligned} U^{(in)}_{p_1,p_2}(x_1', x_2', z') = \exp(-ikz') \prod_{j=1}^{2} \Bigg\{ \sqrt{\frac{q_{0j}'}{q_j'(z'-\zeta_j)}} \left[i\frac{|q_j'(z'-\zeta_j)|}{q_j'(z'-\zeta_j)} \right]^{p_j} \\ \times H_{p_j}\left[\frac{kw_{0j}'(x'-\xi_j)}{\sqrt{2}|q_j'(z'-\zeta_j)|}\right] \exp\left[-i\frac{k}{2}\frac{(x'-\xi_j)^2}{q_j'(z'-\zeta_j)}\right]\Bigg\}. \end{aligned} \tag{8}$$

It is characterized by two complex parameters

$$q_j'(z'-\zeta_j) = z' - \zeta_j + i\zeta_{0j}', \quad j = 1, 2, \tag{9}$$

corresponding to two different locations $z' = \zeta_j$ ($j = 1, 2$) of the waist dimensions in the principal x_j'-directions. $q_j'(0) = q_{0j}' = i\zeta_{0j}'$ ($j = 1, 2$) are the waist position complex parameters, where $\zeta_{0j}' = \frac{kw_{0j}'^2}{2}$ ($j = 1, 2$) are the Rayleigh ranges of the beam and w_{0j}' ($j = 1, 2$) are the fundamental mode waist dimensions of the beam. In expression (8) H_{p_j} (%) are Hermite polynomials of p_j-th order. In any transverse cross-section of the beam the zeros of the Hermite polynomials form an orthogonal net of zero intensity (amplitude) lines, known as mode lines, which widen the amplitude profile in a rectangular spot of sides [19]

$$2w_j'(z'-\zeta_j)_{p_j} = 2\sqrt{2p_j+1}\,w_j'(z'-\zeta_j) = 4\sqrt{2p_j+1}\,\frac{|q_j'(z'-\zeta_j)|}{kw_{0j}'}\,. \tag{10}$$

The net of mode lines consists of p_1 lines parallel to the x_2-axis and p_2 lines parallel to the x'-axis.

4. Diffracted wave field

At a distance $(z-z')$ from the phase CH, the field is defined by the Frensel–Kirchhoff integral

$$\begin{aligned} U^{(\mathrm{diff})}(x_1,x_2,z) = {} & \frac{ik\exp[-ik(z-z')]}{2\pi(z-z')} \int\int_D T(x_1',x_2') \\ & \times U^{(\mathrm{in})}(x_1',x_2',z')\exp\left[-i\,\frac{k}{2}\,\frac{(x_1'-x_1)^2+(x_2'-x_2)^2}{(z-z')}\right]dx_1'dx_2', \end{aligned} \tag{11}$$

x_j ($j = 1, 2$) are rectangular coordinates in the observation plane while $T(x_1', x_2')$ and $U^{(\mathrm{in})}(x_1', x_2', z')$ are given by (6) and (8), respectively. Their insertion in the diffraction integral (11) yields

$$\begin{aligned} U^{(\mathrm{diff})}(x_1,x_2,z) = {} & \frac{ik\exp[-ik(z-\alpha)]}{2\pi(z-z')} \sum_m \sum_\mu i^{\mu-m} J_{\mu-2m}(k\beta) J_m(k\gamma) \prod_{j=1}^{2} \sqrt{\frac{q_{0j}'}{q_j'(z'-\zeta_j)}} \\ & \times \left[i\,\frac{|q_j'(z'-\zeta_j)|}{q_j'(z'-\zeta_j)}\right]^{p_j} \exp\left\{-i\,\frac{k}{2}\left[\frac{x_j^2}{(z-z')} + \frac{\xi_j^2}{q_j'(z'-\zeta_j)}\right] Y_j^{(\mu)}\right\} \end{aligned} \tag{12}$$

where

$$Y_j^{(\mu)} = \int_{-\infty}^{\infty} H_{p_j}\left[\frac{kw_{0j}'(x_j'-\xi_j)}{\sqrt{2}|q_j'(z'-\zeta_j)|}\right] \exp\left\{-i\,\frac{k}{2}\left[A_j^{(\mu)} x_j'^2 - 2B_j x_j'\right]\right\} dx_j', \tag{13}$$

with

$$A_j^{(\mu)} = \frac{1}{q_j'(z'-\zeta_j)} + \frac{1}{z-z'} - \mu\,\frac{\lambda}{r_0^2}, \quad j = 1,\ 2 \tag{14}$$

and

$$B_j = \frac{\xi_j}{q_j'(z'-\zeta_j)} + \frac{x_j}{z-z'}, \quad j = 1,\ 2. \tag{15}$$

The solution of the integral (13) (see Appendix) is:

$$\begin{aligned} Y_j^{(\mu)} = {} & \sqrt{\frac{2\pi}{ikA_j^{(\mu)}}} \left[\frac{|q_j^{(\mu)}(z-\zeta_j^{(\mu)})|}{q_j^{(\mu)}(z-\zeta_j^{(\mu)})}\,\frac{q_j^{(\mu)}(z'-\zeta_j^{(\mu)})}{|q_j^{(\mu)}(z'-\zeta_j^{(\mu)})|}\right]^{p_j} \\ & \times H_{p_j}\left\{\frac{kw_{0j}^{(\mu)}\left[x_j-\xi_j+\mu\,\frac{\lambda}{r_0^2}\,(z-z')\xi_j\right]}{\sqrt{2}|q_j^{(\mu)}(z-\zeta_j^{(\mu)})|}\right\} \exp\left\{i\,\frac{k}{2}\,\frac{B_j}{A_j^{(\mu)}}\right\}, \quad j = 1,\ 2. \end{aligned} \tag{16}$$

With $A_j^{(\mu)}$ and B_j given by (14) and (15), this solution inserted in (12), defines the diffracted wave field by

$$U^{(\mathrm{diff})}(x_1, x_2, z) = \exp[-ik(z-\alpha)\sum_m \sum_\mu i^{\mu-m} J_{\mu-2m}(k\beta) J_m(k\gamma)$$
$$\times \prod_{j=1}^{2} \sqrt{\frac{q_{0j}}{q_j'(z-\zeta_j)} \frac{q_j^{(\mu)}(z'-\zeta_j^{(\mu)})}{q_j^{(\mu)}(z-\zeta_j^{(\mu)})}}$$
$$\times \left[i \frac{|q_j'(z'-\zeta_j)|}{q_j'(z'-\zeta_j)} \frac{q_j^{(\mu)}(z'-\zeta_j^{(\mu)})}{|q_j^{(\mu)}(z'-\zeta_j^{(\mu)})|} \frac{|q_j^{(\mu)}(z-\zeta_j^{(\mu)})|}{q_j^{(\mu)}(z-\zeta_j^{(\mu)})} \right]^{p_j}$$
$$\times H_{p_j}\left[\frac{k w_{0j}^{(\mu)}(x_j - \xi_j^{(\mu)})}{\sqrt{2}|q_j^{(\mu)}(z-\zeta_j^{(\mu)})|} \right] \exp\left\{ -i\frac{k}{2} \left[\frac{(x_j-\xi_j^{(\mu)})^2}{q_j^{(\mu)}(z-\zeta_j^{(\mu)})} - \psi_j^{(\mu)} \right] \right\} \tag{17}$$

where

$$\xi_j^{(\mu)} = \xi_j \left[1 - \mu \frac{\lambda}{r_0^2}(z-z') \right], \quad j = 1, 2, \tag{18}$$

and

$$\psi_j^{(\mu)} = -\mu[2x_j + (z-z') - \xi_j] \frac{\lambda}{r_0^2} \xi_j, \quad j = 1, 2. \tag{19}$$

5. Discussion of the results

Expression (17) shows that the diffracted wave field is represented by a fan of laser beams of the same p_1, p_2-mode orders, as the input beam

$$U^{(\mathrm{diff})}(x_1, x_2, z) = \exp(ik\alpha) \sum_m \sum_\mu i^{\mu-m} J_{\mu-2m}(k\beta) J_m(k\gamma) U_{p_1 p_2}^{(\mu)}(x_1, x_2, z).$$

The fan consists of beams whose propagation axes are the lines

$$x_j = \xi_j^{(\mu)} = \xi_j \left[1 - \mu \frac{\lambda}{r_0^2}(z-z') \right], \quad j = 1, 2, \tag{20}$$

which lie in the plane

$$\xi_2 x_1 - \xi_1 x_2 = 0.$$

The amplitudes of the output beams, apart from their Hermite–Gaussian profiles, are reduced by the values resulting from the Bessel functions $J_{\mu-2m}(k\beta) J_m(k\gamma)$.

Therefore the phase CH multiplicates the incident beam keeping its mode order unchanged, but each of the output beams with $|\mu| \neq 0$ is characterized by transformed beam parameters

$$q_j^{(\mu)}(z-\zeta_j^{(\mu)}) = z - \zeta_j^{(\mu)} + i\xi_{0j}^{(\mu)} = z - z' + \frac{q_j'(z'-\zeta_j)}{1 - \mu \frac{\lambda}{r_0^2} q_j'(z'-\zeta_j)}, \quad j = 1, 2. \tag{21}$$

Their waists in the x_j-directions are found on distances

$$z = \zeta_j^{(\mu)} = z' - \frac{(z'-\zeta_j) - \mu \frac{\lambda}{r_0^2}[(z'-\zeta_j)^2 + \zeta_{0j}'^2]}{\left[1-\mu\frac{\lambda}{r_0^2}(z'-\zeta_j)\right]^2 + \left[\mu\frac{\lambda}{r_0^2}\zeta_{0j}'\right]^2}, \quad |\mu| \neq 0,$$

$$\zeta_j^{(0)} = \zeta_j, \qquad \mu = 0. \qquad j = 1,\ 2, \qquad (22)$$

The Rayleigh ranges of the $|\mu|$-th diffracting order are

$$\zeta_{0j}^{(\mu)} = \frac{k}{2}[w_{0j}^{(\mu)}]^2 = \frac{k}{2}\frac{w_{0j}'^2}{\left[1-\mu\frac{\lambda}{r_0^2}(z'-\zeta_j)\right]^2 + \left[\mu\frac{\lambda}{r_0^2}\zeta_{0j}'\right]^2}, \qquad j = 1,\ 2, \qquad (23)$$

$$\zeta_{0j}^{(0)} = \zeta_{0j}',$$

and their magnifications in the two separate x_j-directions are given by the ratios

$$\frac{w_{0j}^{(\mu)}}{w_{0j}} = \frac{1}{\sqrt{\left[1-\mu\frac{\lambda}{r_0^2}(z'-\zeta_j)\right]^2 + \left[\mu\frac{\lambda}{r_0^2}\zeta_{0j}'\right]^2}}, \quad j = 1,\ 2. \qquad (24)$$

Relations (21), (22) and (24) can be interpreted as typical for a laser beam transfer by a thin spherical lens of focal distance [20], [21]

$$f^{(\mu)} = \frac{r_0^2}{\mu\lambda} \qquad \mu = \pm 1,\ \pm 2,\ \pm 3\dots . \qquad (25)$$

The positive μ-integers correspond to a positive convex thin lens while the negative ones to the concave (negative) thin lens. Let us notice that when

$$z - z' = f^{(\mu)}, \qquad \xi_j^{(\mu)} = 0, \qquad (26)$$

i.e., the μ-th beam axis intersects optical axis of the lens at the μ-th focal distance. And let us outline that for both waist loci

$$f^{(\mu)} \neq \zeta_j^{(\mu)}.$$

The foci are characteristic of the phase hologram, while the waist loci of the output laser beams.

Thus the phase CH with general transmission function (6) behaves like simultaneous existing positive and negative spherical lenses of focal distances (25).

It differs from the Fresnel zone plate [17] by possessing both even and odd foci.

6. Irradiance distribution

The irradiance distribution of the diffracted field is defined by

$$I^{(\mathrm{diff})}(x_1,x_2,z)=\left(\sum_m\sum_\mu i^{\mu-m}J_{\mu-2m}(k\beta)J_m(k\gamma)U^{(\mu)}_{p_1p_2}(x_1,x_2,z)\right)$$
$$\times\left(\sum_{m'}\sum_{\mu'}i^{-(\mu'-m')}J_{\mu'-2m'}(k\beta)J_{m'}(k\gamma)U^{*(\mu)}_{p_1p_2}(x_1,x_2,z)\right). \tag{27}$$

If the observation plane is situated at a distance $z=c$ such that

$$2\sqrt{2p_j+1}\,w^{\mu}_j(c-z')<\sqrt{(\xi^{(\mu+1)}_1-\xi^{(\mu)}_1)^2+(\xi^{(\mu+1)}_2-\xi^{(\mu)}_2)^2}$$
$$=\frac{\lambda}{r_0^2}(c-z')\sqrt{\xi_1^2+\xi_2^2}=\lambda(c-z')\frac{r}{r_0^2},$$

i.e.,

$$c-z'>2\frac{r_0^2}{\lambda}\sqrt{2p_j+1}\,\frac{w^{\mu}_j(c-z')}{r}, \tag{28}$$

the output beams are spatially separated and their spots in the observation plane do not overlap each other.

In this case the irradiance distribution can be represented by the sum of irradiances of the separate beams without taking account of their interference

$$I^{(\mathrm{diff})}(x_1,x_2,z)=\sum_m\sum_\mu J^2_{\mu-2m}(k\beta)J^2_m(k\gamma)I^{(\mu)}_{p_1p_2}(x_1,x_2,z) \tag{29}$$

where

$$I^{(\mu)}_{p_1p_2}(x_1,x_2,z)=\prod_{j=1}^{2}\frac{q'_{0j}}{q_j(z-\zeta_j)}\,\frac{q^{(\mu)}_j(z'-\zeta^{(\mu)}_j)}{q^{(\mu)}_j(z-\zeta^{(\mu)}_j)}$$
$$\times H^2_{p_j}\left[\frac{kw^{(\mu)}_{0j}(x_j-\xi^{(\mu)}_j)}{\sqrt{2}\,|q^{(\mu)}_j(z-\zeta^{(\mu)}_j)|}\right]\exp\left\{-\left[\frac{kw^{(\mu)}_{0j}(x_j-\xi^{(\mu)}_j)}{\sqrt{2}\,|q^{(\mu)}_j(z-\zeta^{(\mu)}_j)|}\right]^2\right\}. \tag{30}$$

In the observation plane $z=c$ the rectangular spots of areas

$$4\prod_{j=1}^{2}\sqrt{2p_j+1}\,w_j(c-\zeta^{(\mu)}_j)=16\prod_{j=1}^{2}\sqrt{2p_j+1}\,\frac{|q^{(\mu)}_j(c-\zeta^{(\mu)}_j)|}{kw^{(\mu)}_{0j}} \tag{31}$$

are distributed (with their sides parallel to each other) around centers:

$$C^{(\mu)}\left\{\xi_1\left[1-\mu\frac{\lambda}{r_0^2}(c-z')\right],\quad \xi_2\left[1-\mu\frac{\lambda}{r_0^2}(c-z')\right]\right\}. \tag{32}$$

Thus the angle deviation of the $|\mu|$-th beam

$$\theta^{(\mu)}=\arctan|\mu|\frac{\lambda}{r_0^2}\sqrt{\xi_1^2+\xi_2^2}=\arctan\left(\mu\frac{\lambda}{r_0^2}r\right),\qquad |\mu|=0,\ 1,\ 2,\ 3\ldots \tag{33}$$

depends on the input coordinates and the μ-th lens power $\frac{1}{f^{(\mu)}}=\mu\frac{\lambda}{r_0^2}$. No deviation occurs when $r = \sqrt{\xi_1^2 + \xi_2^2} = 0$, i.e., when the incidence of the laser beam is on-axial. In this case the beam fan closes into one composite beam, in which each diffracting order has its waists (two real for $\mu > 0$, and two virtual for $\mu < 0$) on positions given by (22). All spots overlap each other, the irradiance distribution includes their interference and is rather complicated.

On the photograph in Fig. 1 an illustration of the fundamental mode spot splitted by the CH, occurring when the incidence is off-axial, is given.

Fig. 1. Fundamental mode spot splitting in the observation plane caused by the phase CH when the incidence is off-axial

The caseof an on-axial incidence fundamental mode laser beam on the same CH is illustrated in Fig. 2.

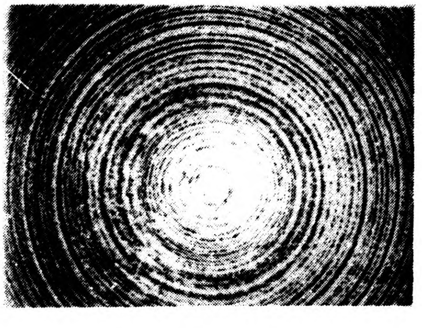

Fig. 2. Diffraction pattern of an on-axial input of a fundamental mode laser beam obtained by phase CH

7. Conclusions

Based on the theoretical results arising from this article, the following conclusions can be formulated:

– The off-axial input of a laser beam on a phase CH is followed by a beam multiplication output of all diffraction orders. There is a fan of beams deviated with respect to the direction of the incident beam (the $\mu = 0$ diffracting order) by the angles given in (33). The fan is in the plane (21). The deviation can be controlled by the off-axial input displacement $r = \sqrt{\xi_1^2 + \xi_2^2}$. Beam multiplication with no deviation occurs when $r = 0$.

– The mode-order of the input beam is not affected by the multiple beam splitting with phase CH.

– Each of the output splitting beams is characterized by two complex parameters which are dependent on the input beam complex parameters by the relations (21). The same relations can be rearranged in the form of Kogelnik ratios [22] as

$$q_j^{(\mu)}(z-\zeta_j^{(\mu)}) = \frac{\left[1-\mu\dfrac{\lambda}{r_0^2}(z-z')\right]q_j'(z'-\zeta_j)+(z-z')}{-\mu\dfrac{\lambda}{r_0^2}q_j'(z'-\zeta_j)+1}, \quad j=1,\ 2. \tag{34}$$

Therefore the Kogelnik ABCD rule of the beam transfer system consisting of the phase CH as an optical element and the free space distance $(z-z')$ expressed in the matrix form is

$$\begin{bmatrix} 1-\mu\dfrac{\lambda}{r_0^2}(z-z') & (z-z') \\ -\mu\dfrac{\lambda}{r_0^2} & 1 \end{bmatrix} = \begin{bmatrix} 1 & (z-z') \\ 0 & 1 \end{bmatrix} \begin{bmatrix} A_j & B_j \\ C_j & D_j \end{bmatrix}, \ j=1,\ 2 \tag{35}$$

where

$$K_j = \begin{bmatrix} A_j & B_j \\ C_j & D_j \end{bmatrix}, \quad j=1,\ 2, \tag{36}$$

are the Kogelnik ABCD matrices which determine the transformation of the complex beam parameters, while

$$T = \begin{bmatrix} 1 & (z-z') \\ 0 & 1 \end{bmatrix} \tag{37}$$

is the translation matrix through the distance $(z-z')$. It is not hard to see from (35) that

$$K_j = T^{-1}\begin{bmatrix} 1-\mu\dfrac{\lambda}{r_0^2}(z-z') & (z-z') \\ -\mu\dfrac{\lambda}{r_0^2} & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -\mu\dfrac{\lambda}{r_0^2} & 1 \end{bmatrix}, \quad j=1,\ 2, \quad |\mu| = 0,1,2,3\ldots. \tag{38}$$

Due to the circular symmetry of the phase CH, there is one Kogelnik matrix for both complex parameters. The Kogelnik matrices (38) characterize the complex beam parameter transformation for both on-axial and off-axial incidence on a phase CH.

— Treating the input and output beam axes as optical rays, we can apply the matrix method to describe the beam deviation caused by a deffracting device as it is our phase CH [23]. If $\begin{bmatrix} \xi_j \\ n' v'_j \end{bmatrix}$ and $\begin{bmatrix} \xi_j^{(\mu)} \\ n v_j \end{bmatrix}$ $j = 1, 2$ are ray vector matrices [24] associated with the beam axes, then for our optical system there exist matrix equations

$$\begin{bmatrix} 1 & (z-z') \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a_j & b_j \\ c_j & d_j \end{bmatrix} \begin{bmatrix} \xi_j \\ n' v'_j \end{bmatrix} = \begin{bmatrix} \xi_j^{(\mu)} \\ n v_j \end{bmatrix}, \quad \begin{matrix} j = 1, 2, \\ |\mu| = 0, 1, 2, 3 \ldots, \end{matrix} \tag{39}$$

n' and n are refractive indices of the input and output media. In our case $n' = n \approx 1$. v'_j and v_j are optical direction tangents of the input and output "rays". $\begin{bmatrix} a_j & b_j \\ c_j & d_j \end{bmatrix} = M_j$ are the direction of ray transfer matrices of the diffracting elements. They are unimodular

$$\det M_j = 1. \tag{40}$$

The unimodularity condition and requirement for the matrix Eq. (39) to be valid for all possible values of ξ_j and v'_j, define the ray transfer matrix by

$$M = \begin{bmatrix} a_j & b_j \\ c_j & d_j \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \frac{v_j - v'_j}{\xi_j} & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \frac{\xi_j - \xi_j^{(\mu)}}{\xi_j (z-z')} & 1 \end{bmatrix}, \quad j = 1, 2, \tag{41}$$

since for normal incidence and $v'_j = 0$ and $v_j = \frac{\xi_j - \xi_j^{(\mu)}}{(z-z')}$ $(j = 1, 2)$ are the input data, while $\xi_j^{(\mu)}$ for a given diffracting device are read from the solution of the diffraction integral. For the case of phase CH they are given by expressions (18). Substituted in the "ray" matrix (41) they give

$$M = \begin{bmatrix} 1 & 0 \\ -\mu \frac{\lambda}{r_0^2} & 1 \end{bmatrix}, \quad j = 1, 2. \tag{42}$$

It turns out that for the phase CH the Kogelnik and "ray" transfer matrices are the same. If we did not choose the coordinate system with its z-axis to coincide with the hologram optical axis, but parallel and displaced at a distance $c = \sqrt{c_1^2 + c_2^2}$, instead of expression (6) for the transmission function we have

$$T(x'_1, x'_2) = e^{ika} \sum_m \sum_\mu i^{\mu - m} J_{\mu - 2m}(k\beta) J_m(k\gamma) \prod_{j=1}^{2} \exp\left[i \frac{k}{2} \mu \frac{\lambda}{r_0^2} (x'_j - c_j)^2 \right] \tag{43}$$

and then it is not hard to see that

$$\xi_j^{(\mu)} = \xi_j - \mu \frac{\lambda}{r_0^2}(z-z')(\xi_j - c_j), \quad j = 1, 2. \tag{44}$$

In this case the "ray" transfer matrices are

$$M_j = \begin{bmatrix} 1 & 0 \\ -\frac{\mu\lambda}{r_0^2}\left(1-\frac{c_j}{\xi_j}\right) & 1 \end{bmatrix}, \quad j = 1, 2, \tag{45}$$

and they differ from the Kogelnik matrices.

Matrices (38) and (45) obtained from our theoretical results (34) and (44) can serve as an "identification card" of the phase CH used as an optical element in a given transfer system.

Appendix

To solve the integrals (13) we make the change of variables ($x_j \to t_j$) by

$$x_j = \frac{B_j}{A_j^{(\mu)}} + \sqrt{\frac{\pi}{kA_j^{(\mu)}}}\, t_j, \quad j = 1, 2, \tag{A.1}$$

introduce the shorter notations

$$\alpha_j^{(\mu)} = \frac{kw'_{0j}}{\sqrt{2}|q'_j(z'-\zeta_j)|}\sqrt{\frac{\pi}{kA_j^{(\mu)}}} \text{ and } \beta_j^{(\mu)} = \frac{kw'_{0j}}{\sqrt{2}|q'_j(z'-\zeta_j)|}\left(\frac{B_j}{A_j^{(\mu)}} - \xi_j\right), \; j = 1, 2, \tag{A.2}$$

and use the Hermite polynomial addition theorem [25]

$$H_p(\alpha t + \beta) = \frac{1}{\sqrt{2^p}} \sum_{l=0}^{p} \binom{p}{l} H_l(\alpha\sqrt{2}t) H_{p-l}(\beta\sqrt{2})$$

to get

$$Y_j^{(\mu)} = \sqrt{\frac{\pi}{kA_j^{(\mu)}}} \exp\left\{i \frac{k}{2} \frac{B_j^2}{A_j^{(\mu)}}\right\} \frac{1}{\sqrt{2^{p_j}}} \sum_{l=0}^{p_j} \binom{p_j}{l} H_{p_j - l}(\beta_j^{(\mu)}\sqrt{2})\, \phi_{l_j}^{(\mu)}, \quad j = 1, 2 \tag{A.3}$$

where

$$\phi_{l_j}^{(\mu)} = \int_{-\infty}^{\infty} H_{l_j}(\alpha_j^{(\mu)}\sqrt{2}t_j) \exp\left(-i\frac{\pi}{2} t_j^2\right) dt_j, \quad j = 1, 2. \tag{A.4}$$

Further integration goes as follows:

For $l_j = 0$

$$\phi_0 = \int_{-\infty}^{\infty} \exp\left(-i\frac{\pi}{2}t_j^2\right)dt_j = \frac{\sqrt{2}}{i}. \tag{A.5}$$

For $l_j = 1$, taking into account that $H_1(\alpha_j\sqrt{2}t_j) = 2\alpha_j\sqrt{2}t_j$,

$$\begin{aligned}\phi_1 &= \int_{-\infty}^{\infty} H_1(\alpha_j^{(\mu)}\sqrt{2}t_j)\exp\left(-i\frac{\pi}{2}t_j^2\right)dt_j \\ &= -\left(2\alpha_j^{(\mu)}\frac{\sqrt{2}}{i\pi}\right)\int_{-\infty}^{\infty} d\left[\exp\left(-i\frac{\pi}{2}t_j^2\right)\right] = 0.\end{aligned} \tag{A.6}$$

To solve the higher l_j-index integrals, we employ the recurrence relations of the Hermite polynomials [26]

$$\frac{dH_j(x)}{dx} = 2lH_{l-1}(x) \quad \text{and} \quad H_l(x) = 2xH_{l-1}(x) - 2(l-1)H_{l-2}(x),$$

and apply the intergration by parts to get

$$\phi_{l_j}^{(\mu)} = \begin{cases} \dfrac{(2s_j)!}{s_j!}\sqrt{\dfrac{2}{\mathrm{i}}}\left[\dfrac{4\alpha_j^{(\mu)}}{i\pi} - 1\right]^{s_j} & l_1 = 2s_j, \\ 0 & l_1 = 2s_j+1, \end{cases} \qquad j = 1,\ 2. \tag{A.7}$$

If we change the notation $(\alpha_j \to \delta_j)$ by putting

$$[\delta_{l_j}^{(\mu)}]^2 = \frac{i\pi}{2i\pi - 4[\alpha_j^{(\mu)}]^2} \qquad j = 1,\ 2, \tag{A.8}$$

we can use multiplication $(\times)$ theorem for the Hermite polynomials [18]

$$H_p(\delta x) = p!\delta^p \sum_{s=0}^{[p/2]} \frac{H_{p-2s}(x)}{s!(p-2s)!}\left[1 - \frac{1}{\delta^2}\right]^s,$$

and get

$$Y_j^{(\mu)} = \sqrt{\frac{2\pi}{ikA_j^{(\mu)}}}\left(\frac{1}{\delta_j^{(\mu)}\sqrt{2}}\right)^{p_j}\exp\left\{i\frac{k}{2}\frac{B_j^2}{A_j^{(\mu)}}\right\}H_{p_j}(\delta_j^{(\mu)}\beta_j^{(\mu)}\sqrt{2}). \tag{A.9}$$

By defining the complex parameters

$$q_j^{(\mu)}(z - \zeta_j^{(\mu)}) = z - z' + \frac{q_j'(z' - \zeta_j)}{1 - \dfrac{\mu\lambda}{r_0^2}q_j'(z' - \zeta_j)}, \qquad j = 1,\ 2,$$

and (A.10)

$$q_j^{(\mu)}(z' - \zeta_j^{(\mu)}) = \frac{q_j'(z' - \zeta_j)}{1 - \dfrac{\mu\lambda}{r_0^2}q_j'(z' - \zeta_j)}, \qquad j = 1,\ 2,$$

we can express $\beta_j^{(\mu)}$ in the following way:

$$\beta_j^{(\mu)} = \frac{q_j^{(\mu)}(z' - \zeta_j^{(\mu)})}{|q_j^{(\mu)}(z' - \zeta_j^{(\mu)})|} \frac{|q_j^{(\mu)}(z - \zeta_j^{(\mu)})|}{q_j^{(\mu)}(z - \zeta_j^{(\mu)})} \frac{kw_j^{(\mu)}(x_j - \xi_j^{(\mu)})}{\sqrt{2}|q_j^{(\mu)}(z - \zeta_j^{(\mu)})|}, \quad j = 1,\ 2 \tag{A.11}$$

where

$$w_j^{(\mu)} = \frac{w'_{0j}}{\left|1 - \frac{\mu\lambda}{r_0^2} q'_j(z' - \zeta_j)\right|}, \quad j = 1,\ 2, \tag{A.12}$$

and

$$\xi_j^{(\mu)} = \xi_j - \frac{\mu\lambda}{r_0^2} \xi_j(z - z'), \quad j = 1,\ 2. \tag{A.13}$$

Thus the argument of the Hermite polynomial in (A.9) can be written as

$$\delta_j^{(\mu)} \beta_j^{(\mu)} \sqrt{2} = \delta_j^{(\mu)} \frac{kw_{0j}^{(\mu)}(x_j - \xi_j^{(\mu)})}{\sqrt{2}|q_j^{(\mu)}(z - \zeta_j^{(\mu)})|}, \quad j = 1,\ 2 \tag{A.14}$$

where

$$\delta_j'^{(\mu)} = \delta_j^{(\mu)} \sqrt{2} \frac{q_j^{(\mu)}(z' - \zeta_j^{(\mu)})}{|q_j^{(\mu)}(z' - \zeta_j^{(\mu)})|} \frac{|q_j^{(\mu)}(z - \zeta_j^{(\mu)})|}{q_j^{(\mu)}(z - \zeta_j^{(\mu)})}, \quad j = 1,\ 2.$$

Since

$$\left[1 - \frac{1}{[\delta_j'^{(\mu)}]^2}\right]^s = \begin{cases} 0 & \text{for} \quad s \neq 0, \\ 1 & \text{for} \quad s = 0, \end{cases}$$

when we apply the multiplication Hermite polynomials theorem for $H_p\left[\delta_j'^{(\mu)} \frac{kw_{0j}^{(\mu)}(x_j - \xi_j^{(\mu)})}{\sqrt{2}|q_j^{(\mu)}(z - \zeta_j^{(\mu)})|}\right]$ and introduce the result in (A.9), we end up with

$$Y_j^{(\mu)} = \sqrt{\frac{2\pi}{ikA_j^{(\mu)}}} \left[\frac{q_j^{(\mu)}(z' - \zeta_j^{(\mu)})}{|q_j^{(\mu)}(z' - \zeta_j^{(\mu)})|} \frac{|q_j^{(\mu)}(z - \zeta_j^{(\mu)})|}{q_j^{(\mu)}(z - \zeta_j^{(\mu)})}\right]^{p_j} H_{p_j}\left[\frac{kw_{0j}^{(\mu)}(x_j - \xi_j^{(\mu)})}{\sqrt{2}|q_j^{(\mu)}(z - \zeta_j^{(\mu)})|}\right]. \tag{A.15}$$

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Круговаа фазовая голограмма как делитель лазерного пучка

Теоретические исследования преобразования пучка общего порядка Эрмита–Гаусса при помощи фазовой голограммы круговых зон показывают его применение в качестве многократного делителя лазерного пучка для внеаксиального падения. Каждый из разделенных составных элементов обладает таким же порядком моды, а также такой же ориентацией как падающий пучок, но они описаны разными комплексными параметрами. Положение талей пучков, а также увеличения локализацией многократных фокусов зонной голограммы, а для данного порядка дифракции исполняют отношения Селфа типичные для преобразования пучка обыкновенной линзой. Матрицы перехода для круговой голограммы (CH) определены на основе правила Когельника ABCD.

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