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Measurement of the angle of rotation using moiré phenomenon in coherent light *

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A double-diffraction grating system for the measurement of the rotation angle is proposed. The method allows an easy change of the measurement sensitivity. The phenomenon theory and the possibilities of the method applications are presented.

1. Introduction

A system of two equal diffraction gratings with lines mutually inclined generates fringes with the period many times larger than the period of the component gratings. The phenomenon known as moiré effect is widely used for different measurements [1]. In the case of angular measurements the radial gratings have been adapted [2]. We propose to apply linear gratings. If a system of two equal diffraction gratings is illuminated by a spherical wave, then a small rotation of one of the gratings can bring, in some angular range, many times larger rotation of moiré fringes. Determining moiré fringes rotation, for example, in the degrees one may measure the grating rotation in minutes or in seconds. The sensitivity of the method can be changed by the change of the ratio of the distance between the gratings to the distance of the point source from the gratings. Any linear shifts of gratings in the direction perpendicular to the rotation axis introduce respective linear shifts of moiré fringes only and they do not influence the measurement of the rotation angle of the grating. The essential imperfection of our proposition is some restraint of the range of the measured angle. In the paper, the phenomenon theory as well as the possibilities of the method applications are presented.

2. Theory

If a diffraction grating G is illuminated by a point source S emitting a monochromatic light of wavelength λ (Fig. 1), then the propagating field behind the grating G appears as a result of the interference of spherical waves generated by

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point sources located on the sphere Σ_F [3]. The centre of this sphere is at M, where the axis SM is perpendicular to the plane of the grating. Although the sphere Σ_F coincides with the source S in the figure it is in the image space of G. This means that the sphere Σ_F is virtual with respect to the grating. The apparent point sources S'_m

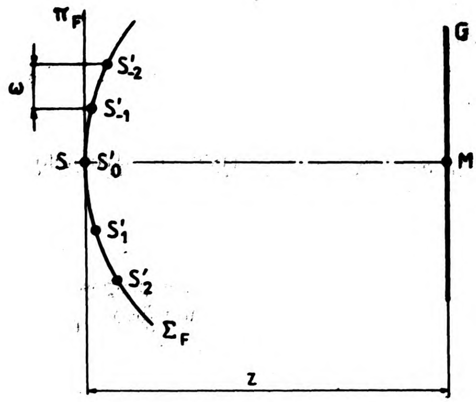


Fig. 1. Propagation of the field emitted by a point source S through a diffraction grating G

($m = 0, \pm 1, \pm 2, \ldots$) are periodically distributed on the sphere Σ_F along the direction perpendicular to the grating lines. The discrete field distribution on Σ_F, as the Fourier transform of the field transmitted through the diffraction grating, is in general not of equiphase type. This is because the Fourier coefficients of the grating transmittance may be complex. Since the relative phase shifts of the light emitted by the apparent sources S'_m will not be essential for our further considerations we can suppose, for simplicity, that these point sources are located in the plane π_F.

According to Figure 1 and the equation of the wave propagation through the diffraction grating, the distance ω between the adjacent points S'_m in the paraxial region can be described by the following expression

$$\omega = \frac{\lambda z}{W} \tag{1}$$

where W is the grating period.

For a system of two gratings G_1 and G_2 (Fig. 2), we can suppose that the second

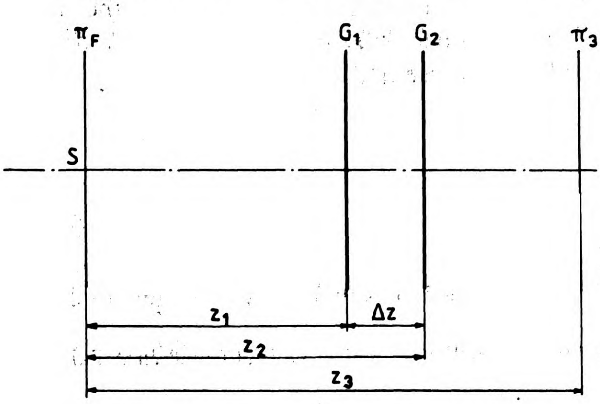


Fig. 2. Double-diffraction grating system with Fourier plane π_F and the plane of observation π_3

grating G_2 is illuminated by a set of apparent point sources of the Fourier distribution of the first grating G_1. Consequently, every point source of the mentioned set generates in the image space of the second grating its own Fourier distribution in the form of equidistant points distributed periodically along the direction perpendicular to the second grating lines. If the lines of both gratings are mutually inclined, then the Fourier spectrum of the field transmitted through two gratings becomes two-dimensional set of points (see [4], for example). The central part of this Fourier distribution is shown in Fig. 3. The vectors $\bar{\omega}_1$ and $\bar{\omega}_2$ are perpendicular to the lines of the first and second diffraction gratings, respectively, and their moduli, according to Fig. 2 and Eq. (1), are given by the expression

$$\omega_j = \frac{\lambda z_j}{W_j}, \quad j = 1,2 \tag{2}$$

where W_1 and W_2 are the periods of the first and second gratings, respectively.

All spots of the spectrum are marked by the labels composed of two numbers m_1, m_2. The spots located on the line covered with the vector $\bar{\omega}_1$ are labelled by $(m_1, 0)$, where $m_1 = 0, \pm 1, \pm 2, \ldots$. They relate to the spectrum of the first grating. The second grating multiplies every spot $(m_1, 0)$ into the set of spots (m_1, m_2), where $m_2 = 0, \pm 1, \pm 2, \ldots$. All spots with $m_1 = \text{const}$ are located on the lines parallel to the vector $\bar{\omega}_2$. The spot $(0,0)$ coincides with the source S.

The field in the image space of a double diffraction gratings system may by treated as a result of the interference of the light emitted by all apparent sources of the Fourier spectrum. Two arbitrarily chosen sources of the plane π_F give in the paraxial region of a plane π_3 (Fig. 2) an intensity distribution in the form of linear fringes perpendicular to the line joining both sources. Analogically to Eq. (2), the period W_f of the fringes fulfills the relation

$$W_f = \frac{\lambda z_3}{\Omega} \tag{3}$$

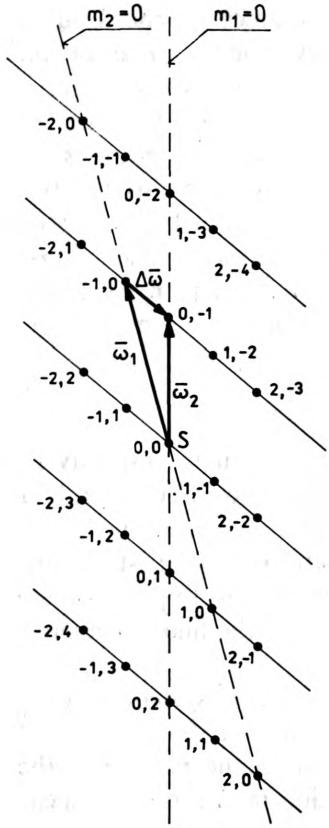
where Ω is the distance between the sources under consideration. In the plane π_F we have many different pairs of sources with different directions and distances. Such situation implies a complex intensity distribution in the plane π_3 as a composition of many linear fringe structures with different directions and periods.

According to Figure 3, if the modulus of the vector

$$\Delta\bar{\omega} = \bar{\omega}_2 - \bar{\omega}_1 \tag{4}$$

is considerably smaller than the moduli of the component vectors $\bar{\omega}_1$ and $\bar{\omega}_2$ ($\Delta\omega \ll \omega_1$ and $\Delta\omega \ll \omega_2$), then the field of fringes may arise with large period with comparison to the periods of other fringe fields. In this case one speaks about moiré fringes or moiré phenomenon.

If we take such a configuration of a diffraction system that the fringes with high frequencies cannot be detected, then in order to explain the angular position of the moiré fringes it is sufficient to analyse three respective spots of the diffraction spectrum shown in Fig. 4. For simplicity it has been supposed that the rotation



▼ Fig. 3. Fourier spectrum of two inclined diffraction gratings illuminated by the point source S

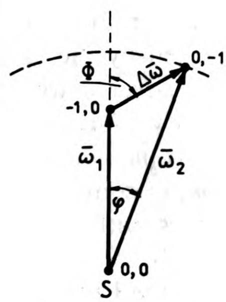


Fig. 4. Central spots of Fourier spectrum of two inclined diffraction gratings

angle φ of the second grating with respect to the first one has to be determined and the lines of the first grating are horizontal (the vector $\bar{\omega}_1$ vertical). Usually diffraction gratings with equal periods are used. In this case $W_1 = W_2$, and because $z_2 > z_1$ (see Fig. 2), then according to Eq. (2) the case fulfilling the relation $\omega_2 > \omega_1$ is more frequent. The moiré fringes are perpendicular to the vector $\Delta\bar{\omega}$. For $\varphi = 0$ (the lines of both gratings are parallel) the direction of moiré fringes is parallel to grating lines. A rotation of the second grating by the angle φ introduces the respective rotation of moiré fringes by the angle Φ (see Fig. 4). It can be clearly seen that in some range of the angle φ we have the relation $\Phi/\varphi \gg 1$. The last relation is the essence of our proposition to determine small angle φ of rotation of the grating G_2 with the aid of the measurement of the angle Φ of the rotation of moiré fringes.

The above statement may be proved analytically. Basing on trigonometric relations we can write

$$\sin\Phi = M(\varphi)\sin\varphi \tag{5}$$

where

$$M(\varphi) = \frac{\omega_2}{\Delta\omega} = \frac{\omega_2}{\sqrt{\omega_1^2+\omega_2^2-2\omega_1\omega_2\cos\varphi}}. \tag{6}$$

Relation (5) can be rearranged to the form in which the coefficient of the proportionality will be less changeable than $M(\varphi)$. For this purpose we denote

$$M_o = \frac{\omega_2}{\omega_2 - \omega_1} \tag{7}$$

and according to (6) we can write

$$M = M_o C, \tag{8}$$

where

$$C = \frac{1}{\sqrt{1 + 4M_o(M_o - 1)\sin^2\frac{\varphi}{2}}}. \tag{9}$$

Note that generally $\Delta\omega \neq \omega_2 - \omega_1$ (see Fig. 4) and $\Delta\omega = \omega_2 - \omega_1$ if $\varphi = 0$ only.

According to Eqs. (5) and (8) it will be $\sin\varphi/2 = \sin\Phi/(2M_0 C \times \cos\varphi/2)$, and after substituting it into (9) and solving this relation with respect to C we obtain

$$C = \cos\Phi\sqrt{1 + A\tan^2} \tag{10}$$

where

$$A = \frac{1}{M_o\cos^2\frac{\varphi}{2}} - \tan^2\frac{\varphi}{2}. \tag{11}$$

Taking into account Eqs. (10) and (8) we can write in place of the relation (5)

$$\tan\Phi = M_o\sqrt{1 + A\tan^2\Phi}\,\sin\varphi. \tag{12}$$

Equation (12) gives us the demanded relation between the angle φ of the rotation of the grating G_2 (see Figs. 2 and 4) and the angle Φ of the rotation of moiré fringes. The quantities M_o and A are defined by Eqs. (7) and (11), respectively.

The function $\tan\Phi$ is strongly nonlinear. This means that the useful range of the angle Φ is $|\Phi| < 60°$. Because the quantity M_o is many times larger than unity the range of the measured angle φ is small and it decreases with the increasing of the value of M_o. Moreover, for sufficiently large value of M_o, according to (11), the values A are small and we can write

$$\tan\Phi = M_o\sin\varphi \tag{13}$$

or even

$$\tan\Phi = M_o\varphi. \tag{14}$$

Relation (12) is presented in Fig. 5 for $M_o = 100$. The broken straight line concerns a linear relation $\Phi = M_o\varphi$. The curves for other values of M_o are similar, but they have different ranges of the angle φ (see Table).

The change of the quantity $\Phi/(\varphi M_o)$ as a function of the angle Φ calculated numerically is shown in Fig. 6. It allows the presentation of the influence of the coefficient M_o in the nonlinear relation (12). Practically the function does not change for $M_o > 100$.

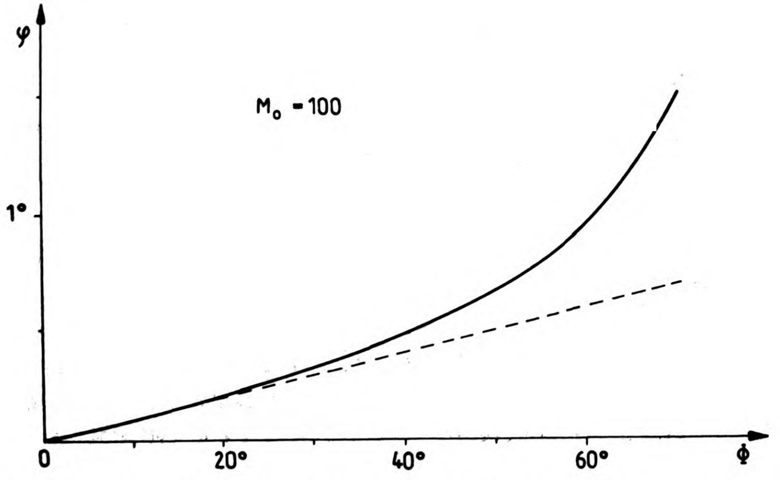


Fig. 5. Angle of the grating rotation φ as the function of the moiré fringe rotation Φ for $M_o = 100$ (see Eq. (12))

Ranges of the measured angle φ for $|\Phi|_{max} = 60°$

M_o	10	100	1000	10000
φ	8.8°	59′	6′	36″

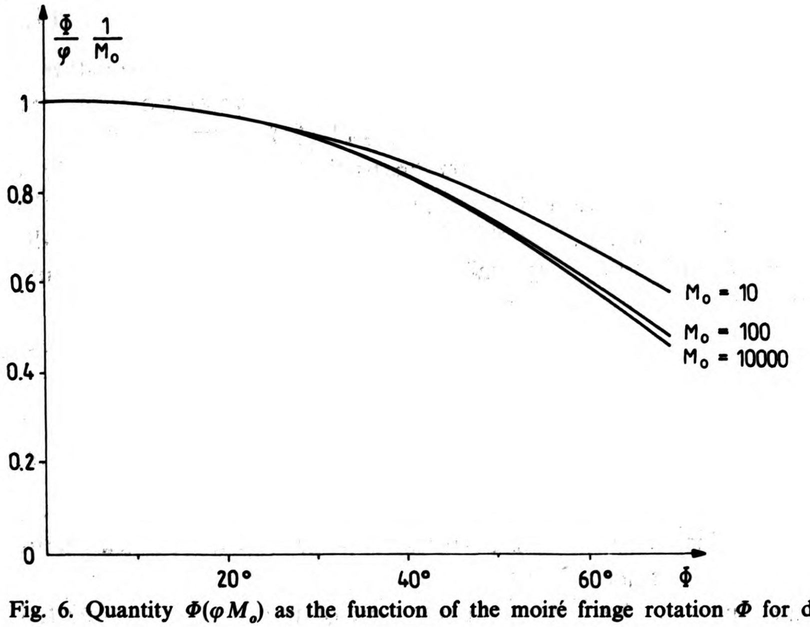


Fig. 6. Quantity $\Phi(\varphi M_o)$ as the function of the moiré fringe rotation Φ for different values of M_o

The change of the measurement sensitivity in the different ranges of the angle Φ is given, according to (13), by

$$\frac{\Delta\Phi}{\Delta\varphi M_o} = \cos^2\Phi\cos\varphi. \tag{15}$$

We have taken the simplified expression (13) in place of (12), because the influence of the term under the root is negligible. The relation (15) is presented in Fig. 7 for two

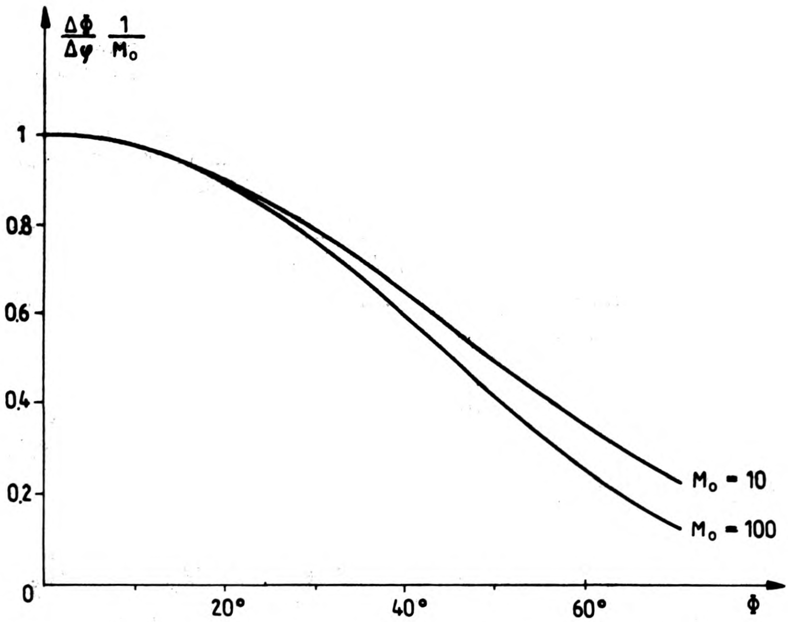


Fig. 7. Quantity $\Delta\Phi(\Delta\varphi M_o)$ as the function of the moiré fringe rotation Φ for two values of M_o

values of M_o. For $M_o \geqslant 100$, because of small value of φ we can put $\cos\varphi \approx 1$ and the right hand side of the relation (15) becomes independent of M_o. According to Fig. 7 we can conclude that with the increase of the angle Φ the sensitivity of the measurement decreases and for $\Phi = 60°$ it is 3 times smaller than for $\Phi = 0$.

3. Design problems

A system allowing the measurement of angles of the rotation with the aid of moiré fringes is shown in Fig. 8. An objective O focuses the light beam generated by a laser L in the plane of a pinhole P. If the angle of divergence of the laser beam behind the pinhole is sufficiently large we can conclude that diffraction gratings G_1 and G_2 are illuminated by the point source P. The moiré fringes are observed on a diffusing screen S. If the lines of both gratings are parallel in the horizontal direction ($\varphi = 0$, $\Phi = 0$, see Fig. 4) then the moiré fringes are horizontal as shown in Fig. 9a.

A rotation of the grating G_2 by the angle introduces the respective rotation of the moiré fringes by the angle Φ (see Figs. 4 and 9**b**). The increase of the spatial frequency of moiré fringes shown in Fig. 9**b** with comparison to fringes in Fig. 9**a** results from the increase of the distance $\Delta\omega$ of the spots $(-1,0)$ and $(0,-1)$ with the increasing of

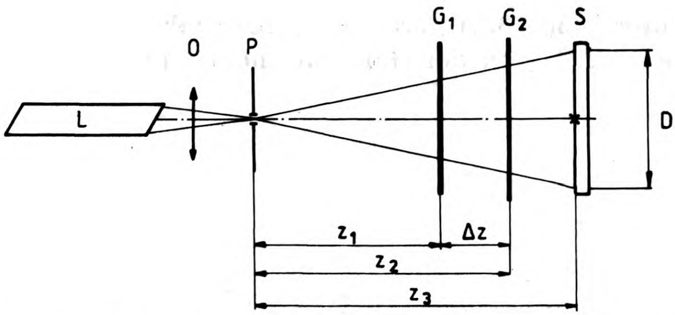


Fig. 8. Optical system applied to the measurement of the angle of the rotation of the grating G_2 or G_1 from the angle of the rotation of moiré fringes on the diffusing screen S

the angle Φ. Now according to relation (12) the measurement of the angle Φ allows the determination of a small rotation of the grating G_2. If on the diffusing screen a cross is marked, then this measurement can be accomplished with the aid of the measurement of the rotation of the screen.

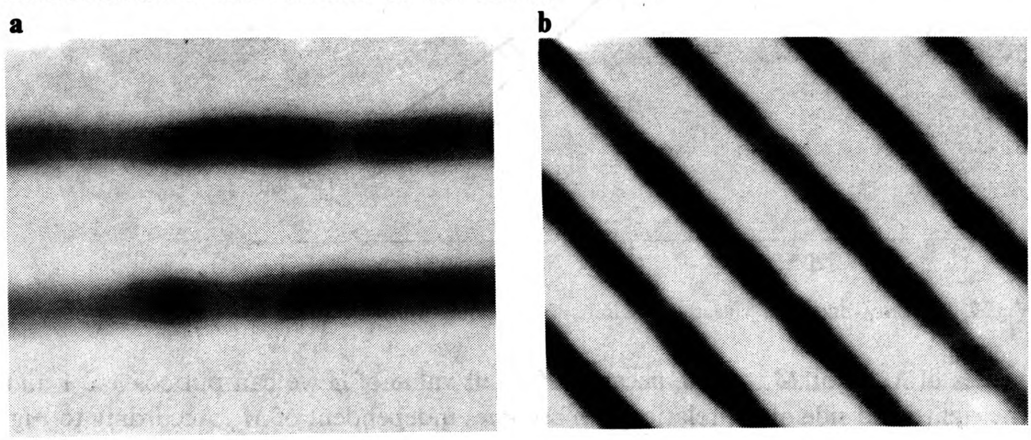


Fig. 9. Photographs of moiré fringes obtained in the configuration shown in Fig. 8 for two different angles: **a** – $\varphi = 0$, **b** – $\varphi \neq 0$

Both types of gratings, phase and amplitude, can be used. Moreover, the gratings do not have to be equal. For the specified gratings it is necessary to assort the configuration of the system in such a manner that the fringes contrast will be sufficiently high and the coefficient M_o will have a proper value.

Substituting Eq. (2) into (7), because $z_2 = z_1 + \Delta z$ (see Fig. 8), we obtain the following expression

$$M_o = \frac{z_2}{z_2\left(1 - \frac{W_2}{W_1}\right) + \Delta z}, \tag{16}$$

that for the equal periods of both gratings ($W_1 = W_2$) simplifies to the form

$$M_o = \frac{z_2}{\Delta z}. \tag{17}$$

It results from Eq. (16) or (17) that for a given distance Δz between the gratings we can change the value of M_o changing the position of the point source with respect to the gratings (change of z_1 and z_2). But with increasing M_o the period of moiré fringes increases. The increase of the value of M_o has its limit because in the field of view one fringe has to be seen at least.

The most unfavourable situation from the point of view of the number of fringes occurs for $\varphi = 0$ (the parallel lines of gratings). In this case according to Eq. (3) and Fig. 8 the fringe period equals

$$W_{fo} = \frac{\lambda z_3}{\omega_2 - \omega_1}. \tag{18}$$

Taking into account Eqs. (2) and (16) we can write

$$\frac{W_{fo}}{W_2} = M_o \frac{z_3}{z_2}. \tag{19}$$

If the dimension of the field of view is D then the condition $W_{fo} < D$ has to be fulfilled. This means, according to (19), that for large value of M_o the gratings with small period have to be applied.

The precision of the measurement of the angle Φ with the aid of the screen rotation depends on many factors such as the number of fringes in the field of view, the angular dimension of one fringe, the contrast of moiré fringes, the image intensity and others. The analysis of this problem is out of the scope of our considerations. We have found only that for binary amplitude gratings with typical conditions in an optical laboratory (the dimension of the screen – 40 mm, visual observation) it is not difficult to obtain $\Delta\Phi \approx 1°$ for one fringe in the field of view and $\Delta\Phi \approx 15'$ for 10 fringes. According to (15) the error $\Delta\varphi$ relates to the error $\Delta\Phi$ by the expression

$$\Delta\varphi = \frac{\Delta\Phi}{M_o \cos^2\Phi} \tag{20}$$

($\cos\varphi \approx 1$). For $\varphi = 0$, since $\Phi = 0$, $\Delta\varphi = \Delta\Phi/M_o$ only.

The change of the fringe contrast has not the essential influence on the precision of the measurement of the angle Φ, but it is necessary to avoid such configuration for which the fringe contrast is very small.

The optimum position of the diffusing screen is connected with a plane of the maximum contrast of moiré fringes. The position of such plane depends not only on the geometrical configuration of the system but also on the types of the gratings [5], [6].

For amplitude gratings the case with maximum contrast occurs when the self-images of both gratings arise in the plane of the screen simultaneously. Denoting by $z_{s,K}$ the distance from the point source to K-th self-image of the first grating (see Fig. 10) we can write [3]

$$\frac{1}{z_{s,K}} = \frac{1}{z_1} - \frac{2W_1^2}{z_1^2 \lambda} K, K = 1, 2, \dots . \tag{21}$$

For simplicity the real self-images have been taken into account only. We can

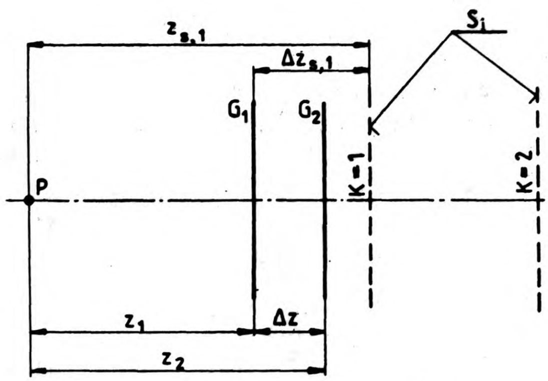


Fig. 10. Exemplary positions of the self-images $S_i(K = 1, 2)$ of the grating G_1 in the double-diffraction grating system

rearrange Eq. (21) introducing the distance $\Delta z_{s,K}$ of the K-th self-image of the first grating from this grating. Because $\Delta z_{s,K} = z_{s,K} - z_1$ then in place of Eq. (21) we get

$$\Delta z_{s,K} = \frac{\Delta z_s K}{1 - \dfrac{\Delta z_s K}{z_1}} \tag{22}$$

where

$$\Delta z_s = \frac{2W_1^2}{\lambda} \tag{23}$$

is the distance between the adjacent self-images of the grating in the case of plane wave illumination ($\Delta z_s = \Delta z_{s,K} - \Delta z_{s,K-1}$ for $z_1 \to \infty$). In Fig. 10 two first self-images ($K = 1, 2$) of the first grating are marked only. The analogical relations can be written for the second grating. The self-images of the second grating are not marked in Fig. 10.

The condition of the coincidence of the self-images of both gratings in the screen plane may conduce to the case with a large distance between the screen and the gratings. Moreover, the change of the measurement sensitivity by the change of the source position (the change of the distances z_1 and z_2 simultaneously) causes different changes of the positions of the sets of the self-images of both gratings. According to Eq. (22) these changes increase with the increasing of the self-image numbers. For this reason it is better to resign from the condition of the maximum contrast and to adjust the gratings and the screen as near as possible. Such configuration allows the design of a compact measuring instrument. If a grating with small period is used, because in this case the distances between the self-images are small, one can put the second grating in the plane of the first self-image of the first grating, and the screen as previously near the second grating.

It is worth to emphasize that one can find disadvantageous elements adjusting with the zero contrast of moiré fringes. The positions of the elements in such a case depend on the type of gratings. For example, for Ronchi gratings (the binary gratings with equal transparent and opaque lines it is necessary to avoid the positions of the screen determined by Eq. (22) with $K = 1/2, 3/2, 5/2, \ldots$. It concerns the screen positions with respect to both the first and second gratings. If the changes of the measurement sensitivity are so large that the cases with zero contrast are possible, one can equip the instrument with the displacement of the screen.

4. Examples of the applications

Our proposition of angular measurements has the following properties:

i) limited range of measured angles,
ii) nonlinear scale,
iii) facility of the changes of the measurement sensitivity,
iv) insensibility to linear grating displacement.

The proposed method can be applied to the measurements of the angle of the rotation of any body particularly in the cases when it is not possible to eliminate accidental displacements of the body during the measurement. Undoubtedly a small range of measured angles hardly limits the possibility of the applications, moreover the nonlinear scale requires the proper calibration. On the other hand, it seems that our proposition can be fitted perfectly to various adjustments.

Let a diffraction G_2 be fixed to an element which must keep a constant angular position during an operation undefined here (Fig. 11). If this operation introduces

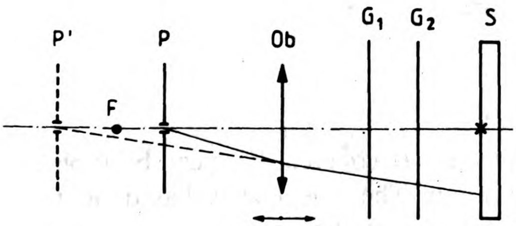


Fig. 11. Change of the measurement sensitivity by an axial displacement of an objective *Ob*

any rotation of the element we can detect it by observing the respective rotation of moiré fringes and next turn the element to the previous position. To make it possible it is necessary to set an initial position of moiré fringes before the operation. If the grating G_2 with the element is in the starting position, then by turning the grating G_1 we go to the position, where the period of moiré fringes is the largest. We begin to perform this adjustment with the lowest sensitivity. In order to facilitate the change of the measurement sensitivity an objective *Ob* is added which images the point source *P* into the point *P'*. The displacement of *Ob* along the optical axis allows an easy change of the distance between the gratings and the point source *P'* in a large range. In the limit case, when the object focus F of the objective Ob coincides with the point P, the image P' is at infinity and theoretically the measurement sensitivity becomes infinitely large. For the lowest sensitivity the objective has to be near the point P, then the distance between the gratings and the point P' is the smallest. After the adjustment of moiré fringes in this position we repeat this procedure displacing step by step the objective to the direction of the higher sensitivities. The real position with the highest sensitivity concerns moiré fringes with one fringe in the field of view. The procedure of the initial adjustment of the measurement system is accomplished when the displacement of the objective Ob does not introduce any perceptible rotation of moiré fringes. Now after any operation performed on the element with the grating G_2 it is sufficient to turn this element in such a way that moiré fringes come back to the initial position for the highest sensitivity. The procedure of the return to the constant angular position of the element can begin with the low

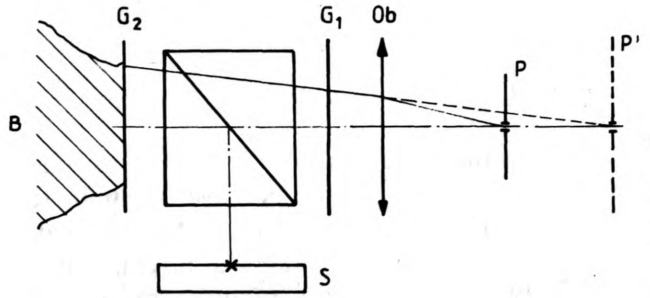


Fig. 12. Measurement of the angle of the rotation of a body B with the aid of a reflecting diffraction grating G_2

sensitivity of the measurement. It facilitates the determination of the direction of the rotation.

Figure 12 presents the application of our method to the determination of the rotation of a body B with a reflecting grating G_2.

5. Summary

It has been pointed out that the moiré phenomenon in coherent light can be applied to the measurement of the angle of the rotation. The principle is based on the rotation of moiré fringes with the rotation of one of the gratings in a double-diffraction grating system. The method allows an easy change of the measurement sensitivity which is particularly convenient in different angular adjustments.

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Измерение угла вращения при помощи явления линий мори в когерентном свете

Предложено применение двойной дифракционной решётки для измерения угла вращения. Метод способствует лёгкому изменению измерительной чувствительности. Дана феноменологическая теория а также возможности применения этого метода.