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MORE ON ONE-COMMODITY MARKET GAMES

In this paper we present a new definition related to *super-balanced games* interpreted as a particular type of cooperative game with transferable utility (TU). A concept of solution is defined and an existence result is determined. The fundamental aim of the paper is to extend the results previously obtained by the same author.

Keywords: *TU cooperative games, super-balanced coalition*

1. Introduction

Cooperative game theory has been frequently used to model various economic scenarios. As is well-known, economic systems have been treated as cooperative games, with the core of an economy commonly used as the main concept of solution. Some specific economic scenarios have been modeled within a cooperative/competitive framework. Shapley and Shubik [9] derived the "market game" from an exchange economy. "Production games" are used to study production problems in a competitive environment [7]. Cost allocation problems have also been modeled as cooperative games by several authors [1], [4] and [5]. Our approach concerns a particular market with several agents who trade one commodity. Each bundle of the commodity sold by an agent has a social value which includes all the relevant investments, costs and expected profit. The agent should trade a bundle of his commodity at a price which reflects this value. Here, we focus on the problem of the stability of prices when they are negotiated by super-balanced coalitions. This paper develops some results previously obtained by the same author [3].

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2. Notation and definitions

Let be $N = \{1, 2, ..., n\}$ the set of agents. R_+^n denotes the positive orthant of n-dimensional Euclidean space. If $x = (x_1, ..., x_n)$, $y = (y_1, ..., y_n) \in R^n$ we shall write $x \geq y$, if $x_i \geq y_i$ for all $i = 1, 2, ..., n$, and $x > y$, if $x \geq y$, but $x \neq y$. If $x \in R_+^n$ and $S \subseteq \{1, 2, ..., n\}$, one denotes the n-vector whose i-th component is x_i if $i \in S$ and 0 if $i \notin S$ by x_S. Finally, let $\overline{S}$ denote the complement of S with respect to $\{1, 2, ..., n\}$. Let G be the family of subsets of the set $N = \{1, 2, ..., n\}$.

Definition 1. G is said to be *super-balanced* if there exists a system of positive weights $(w_S)_{S\in G}$ such that

$$\sum_{S\in G, i\in S} w_S \geq 1, \ \forall i \in N. \tag{1}$$

Let us consider $G = \{S_1, ..., S_n\}$ as a collection of non-empty subsets of $N = \{1, 2, ..., n\}$. A family S is said to be balanced if there exists a system of positive weights $(w_S)_{S\in G}$ such that $\forall i \in N, \sum_{S\in G, i\in S} w_S = 1$.

Remark 1. A super-balanced family of coalitions is simply a covering of the grand coalition N.

3. The positive core of a TU cooperative game

Starting from classical results in Game Theory, it is known that a transferable utility (TU) cooperative game can be represented by the pair (N,a), where a: $2^N \to R$ is the characteristic function. The properties required for a depend on the interpretation of the game. To analyse various economic scenarios, it might be useful to consider $a(S)$ to be the maximum total revenue of the players in S, but alternatively we can interpret $a(S)$ as the minimum total cost (penalty) incurred by S.

For this latter case, it is natural and convenient to reverse the inequalities traditionally used in cooperative game theory. Therefore, the core $C(N, a)$ of the game (N, a) will be defined as

$$C(N, a) = \left\{ y \in R^n \mid \sum_{i\in S} y_i \leq a(S), \forall S \subset N, \sum y_i = a(N) \right\}. \tag{2}$$

From the Bondareva–Shapley Theorem [2], [8], the core is non-empty if and only if the game is balanced. In this paper we study the case in which there exist super-balanced coalitions.

Definition 2. A game is said to be *super-balanced*, if for every super-balanced family $G \subseteq 2^N$, and for every associated system of weights $(w_S)_{S \in G}$ one has

$$\sum_{S \in G} w_S a(S) \geq a(N) . \tag{3}$$

4. The model

We want to consider the realistic situation of a market in which several agents trade the same type of commodity. Each agent sells the amount of the commodity demanded by his consumers. In principle, each consumer has his own preferred agent, but he can change if the prices of others agents become more attractive. Therefore, each agent should satisfy the demand for his produce which is dependent on all prices. Each group of agents (coalition) negotiates the individual prices of its members, in order to equilibrate the value of the good sold with the total amount of money the consumers should pay for it.

Let us introduce the following notation into our model:

A system of prices is any $p \in R_+^n$. If $p = (p_1, ..., p_n)$, then p_i is the i-th agent's price for a unit of the good. The demand for the good is given by a vector function $d : R_+^n \to R_+^n$. For $p \in R_+^n$, the component $d_i(p)$ represents the demand for the good from the i-th agent given the price vector p. The symbol $\langle .,. \rangle$ denotes the scalar product of two vectors. If total demand is represented by a non-negative vector x, then the value supported by the market is $v(x)$, where $v : R_+^n \to R$ is the value function. The triple $\Gamma = (N, d, v)$ is called the *one-commodity market game* [3]. Now we will define an optimal system of prices $p^* \in R_+^n$ to be one that satisfies the following conditions:

$$\langle p^*, d(p^*) \rangle = v(d(p^*)) . \tag{4}$$

Also, there are no $p \in R_+^n$ and $\varnothing \neq S \subseteq N$ such that

$$p_S < p_S^* ,\ p_{\bar{S}} = p_{\bar{S}}^* , \tag{5}$$

$$d_S(p) \geq d_S(p^*) , \tag{5'}$$

$$\langle p_S, d_S(p) \rangle = v(d_S(p)) . \tag{5''}$$

Condition (4) simply says that the total revenue equals the value of the total demand $v(d(p^*))$ at price p^*. From conditions (5′)–(5″), p^* is stable; no coalition of agents can become more attractive to consumers by reducing their prices while simultaneously maintaining their income from sales.

Definition 3. The pseudo-core of the one-commodity market game $\Gamma = (N, d, v)$ is the set of price vectors satisfying conditions (4)–(5″). This set will be denoted by $PC(\Gamma)$.

The market considered here comes within the framework of imperfect competition. In fact, we know that the agents operating in such an economic framework can decide on the price of their commodity. But when all the agents find an agreement by which it is possible to obtain stability of the market, then we may introduce the concept of a stable system of prices satisfying the above conditions, (4) together with (5), (5′) and (5″). In this case, when these conditions are satisfied the market considered is at equilibrium.

A Case study. Let us consider a Cournot–Bertrand oligopoly, where agent *i* sells the good at price *pi*, $i = 1, 2, ..., n$. The demand for the *i*-th agent's good d_i depends on the price vector $p = (p_1, ..., p_n)$, and the cost function (including a reasonable profit) is linear with respect to demand, i.e. c_i□ $d_i(p)$, where $c_i > 0$. According to our terminology (N, d, v) is a one-commodity market game, where $v : R_+^n \to R$ is defined by $v(x) = \sum_{i \in N} c_i x_i$. From condition (4) it follows that $p^* = (p_1^*, ..., p_n^*)$ is a stable system of prices if

$$\sum_{i \in N} p_i^* d_i(p^*) = \sum_{i \in N} c_i d_i(p^*)$$

and

$$S \subset N,\ p_S < p_S^* \Rightarrow \sum_{i \in S} p_i d_i(p_S, p_{\overline{S}}^*) < \sum_{i \in S} c_i d_i(p_S, p_{\overline{S}}^*).$$

In particular, it follows that

$$\forall i \in N,\ p_i < p_i^* \Rightarrow p_i \cdot d_i(p_i, p_{\overline{\{i\}}}^*) < c_i \cdot d_i(p_i, p_{\overline{\{i\}}}^*).$$

Assuming that the demand functions are strictly positive, it clearly follows that $p^* = (c_1, ..., c_n)$ is a stable system of prices.

Let us introduce the following assumptions useful in proving the central result:

(a) For every $i \in N$, d_i is a continuous, strictly positive and decreasing function, i.e.

$$x > y \Rightarrow d_i(x) < d_i(y),$$

$$d_i \in C[R_+^n],$$

$$d_i(x) > 0.$$

(b) The value function v is continuous, non-negative and increasing, i.e.

$$x > y \Rightarrow v(x) > v(y),$$

$$v \in C[R_+^n],$$

$$v(x) > 0.$$

(c) For every $x \in R_+^n$, for every super-balanced family $G \subseteq 2^N$, and for every associated system of weights $(w_S)_{S \in G}$,

$$\sum_{S \in G} w_S v(x_S) \geq v(x).$$

(d) There exists a positive constant *M*, such that $v(d_i(p))/d_i(p) \leq M$, for every $i \in N$ and all $p \in R_+^n$.

(e) The scalar product $\langle p, d(p) \rangle$, is a non-decreasing function of *p* on R_+^n.

Remark 2. What is actually restrictive is the requirement in assumption (c) that certain inequalities should be satisfied by the associated weights. It is easy to see that, with respect to these inequalities, only minimal systems of weights are interesting.

At this point we can introduce the following:

Theorem 1. Let $\Gamma = (N, d, v)$ satisfy assumptions (a)–(e). Then, $PC(\Gamma) \neq \varnothing$.

The proof of this theorem makes use of the following lemma.

Lemma 1. Let (N, a) be a transferable utility cooperative game with characteristic function $a : 2^N \to R$. Suppose that *a* has strictly positive values and

$$a(N) \leq \sum_{S \in G} w_S a(S) \tag{6}$$

for every super-balanced family *G* and associated system of weights $(w_S)_{S \in G}$.

It follows that the positive core $C^+(N, v) = C(N, v) \bigcap R_+^n$ of the game (N, v) is nonempty.

Proof:

$$C^+(N, a) = \left\{ y \in R_+^n \middle| \sum_{i \in S} y_i \leq a(S), \forall S \subset N, \sum\nolimits_{i \in N} y_i = a(N) \right\}.$$

As in the proof of the Bondareva–Shapley theorem regarding the balanced core, one observes that $C^{+}(N,a)$ consists of all the optimal solutions of the linear programming problem

$$(L): \max\left\{ \sum_{i\in N} y_i \,\middle|\, y \in R_{+}^{n}, \sum_{i\in S} y_i \le a(S), \forall S \subseteq N \right\},$$

provided that the optimal value of (*L*) is $a(N)$.

Let (*D*) be the dual of (*L*), i.e.

$$(D): \min\left\{ \sum_{S\in 2^N} w_S a(S) \,\middle|\, w \in R_{+}^{2^N}, \sum_{i\in S} w_S \ge 1, \forall i \in N \right\}.$$

Since $a(S) > 0, \forall S \subseteq N$, then the problem (*L*) has feasible solutions.

Since $\sum_{i\in N} y_i \le a(N)$, then it has optimal solutions and its maximum value is bounded above by $a(N)$. In order to prove the lemma, it is sufficient to show that the minimum value in (*D*) is exactly $a(N)$.

Setting $w_N = 1$, $w_S = 0, \forall S \ne N$, one obtains a feasible solution of (*D*). Hence, the minimum value of (*D*) does not exceed $a(N)$.

Now, let $w = (w_S)_{S\in 2^N}$ be any feasible solution of (*D*). Set $G = \{S \mid w_S > 0\}$. Then *G* is super-balanced and $(w_S)_{S\in G}$ is a system of weights. From (3) one obtains

$$\sum_{S\in 2^N} w_S a(S) = \sum_{S\in G} w_S a(S) \ge a(N).$$

Therefore, the minimum value of (*D*) is $a(N)$.

Now let us denote $P_M = \{p \in R_{+}^{n} \mid p_i \le M, \forall i \in N\}$.

For every $p \in R_{+}^{n}$ consider the game (N, a_p) whose characteristic function is defined by $a_p(S) = v(d_s(p))$, $\forall S \subseteq N$ and define the correspondence (the set-valued function) φ from P_M to R_{+}^{n} by

$$\varphi(p) = \{q \in R_{+}^{n} \mid (q_i d_i(p))_{i\in N} \in C^{+}(N, a_p)\}.$$

Lemma 2. Let $\Gamma = (N, d, v)$ satisfy assumptions (a)–(d). It follows that φ is a closed correspondence from P_M to itself, with non-empty, compact and convex values.

Proof: Obviously, for $p \in P_M$, $d_S(p) > 0$ and $v(d_S(p)) > v(0) \ge 0$.

If we apply Lemma 1, assumption (c) is satisfied and taking into account that $\varphi(p) \neq \emptyset$ for every $p \in P_M$, if $q \in \varphi(p)$, then $(q_i d_i(p))_{i\in N} \in C^+(N, v_p)$, $q \geq 0$ and $q_i d_i(p) \leq v(d_i(p)), \forall i \in N$ (from the definition of the core). Hence, from (d) it follows that $q \in P_M$, since $\varphi(p)$ is clearly convex.

Let us now show that the correspondence φ is closed.

Consider the convergent sequences $p^k \to p^0$, $(p^k) \subseteq P_M$, $q^k \to q^0$, $q^k \in \varphi(p^k), \forall k$.

For every k one has

$$\sum_{i\in S} q_i^k d_i(p^k) \leq v(d_S(p^k)), \forall S \subset N, \sum_{i\in N} q_i^k d_i(p^k) = v(d(p^k)).$$

From the continuity of d_i and v, it follows that:

$$\sum_{i\in S} q_i^0 d_i(p^0) \leq v(d_S(p^0)), \forall S \subset N, = \sum_{i\in N} q_i^0 d_i(p^0) v(d(p^0)).$$

Since $q^0 \geq 0$, it follows that $q^0 \in \varphi(p^0)$. Finally, the compactness of $\varphi(p)$ follows from the closedness of φ and the compactness of P_M.

Proof of Theorem 1

From lemma 2, the correspondence $\varphi : P_M \to 2^{P_M}$ satisfies all the assumptions of Kakutani's fixed point theorem. Let p^* be any fixed point, i.e. $p^* \in \varphi(p^*)$.

We will show that $p^* \in PC(\Gamma)$.

It follows easily that $p^* \geq 0$ and

$$\langle p^*, d(p^*)\rangle = \sum_{i\in N} p_i^* d_i(p^*) = v(d(p^*)) ,$$

so (4) is satisfied.

Now let us suppose that there exists $p \in R_+^n$ satisfying (5)–(5″), for some S. Since $p < p^*$, it follows that $d_i(p) > d_i(p^*), \forall i \in N$, so

$$\langle p_S, d_S(p)\rangle = v\big(d_S(p)\big) > v(d_S(p^*)) \geq \sum_{i\in S} p_i^* d_i(p^*) = \langle p_S^*, d_S(p^*)\rangle . \tag{7}$$

From assumption (e), it follows that

$$\langle p, d(p)\rangle \leq \langle p^*, d(p^*)\rangle .$$

Firstly, it follows from the monotonicity of $d_i, i \in \overline{S}$, see assumptions (a) and (e), that

$$\langle p, d(p)\rangle = \langle p_{\bar{S}}, d_{\bar{S}}(p)\rangle + \langle p_S, d_S(p)\rangle$$

$$= \langle p^*_{\bar{S}}, d_{\bar{S}}(p)\rangle + \langle p_S, d_S(p)\rangle \geq \langle p^*_{\bar{S}}, d_{\bar{S}}(p^*)\rangle + \langle p_S, d_S(p)\rangle ,$$

and secondly one has

$$\langle p^*, d(p^*)\rangle = \langle p^*_{\bar{S}}, d_{\bar{S}}(p^*)\rangle + \langle p^*_S, d_S(p^*)\rangle .$$

From the last three relations, it follows that

$$\langle p^*_S, d_S(p^*)\rangle \geq \langle p_S, d_S(p)\rangle ,$$

which contradicts (7).

5. Conclusions

A deeper understanding of the concept of *pseudo-core* could result from the following comments. Let us associate the *one-commodity market game* $G = (N, d, v)$ with the non-transferable utility (NTU) game represented by (N, V), where the coalitional function V is defined by

$$V(S) = \{p \in R^n_+ \mid \langle p_S, d_S(p)\rangle \geq v(d_S(p))\} ,$$

if $\varnothing \neq S \subset N$ and

$$V(N) = \{p \in R^n_+ \mid \langle p, d(p)\rangle = v(d(p))\} .$$

We can observe that the *pseudo-core* of G contains the core of (N, V). Thus, in particular, every system of prices belonging to the core of the game (N, V) is stable.

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Więcej o grach rynkowych z jednym dobrem

Teoria gier kooperacyjnych jest często używana do modelowania różnych scenariuszy ekonomicznych. Podstawowym pojęciem w modelowaniu tych scenariuszy jest pojęcie jądra jako rozwiązania takich gier. Niektóre specyficzne scenariusze modelowane były w ramach teorii gier kooperacyjnych/konkurencyjnych. Shapley i Shubik w 1969 roku otrzymali „grę rynkową" jako wynik modelowania ekonomii wymiany; Owen w 1975 roku wprowadził „grę produkcyjną", która może służyć do badania problematyki produkcji w środowisku konkurencji ekonomicznej; zagadnienie alokacji kosztów było z kolei modelowane jako gra kooperacyjna przez różnych autorów (Bendali F. i in. [1], Gambarelli G. [4]) czy Granot D. i in. [5]). Nasze podejście dotyczy specyficznego rynku z kilkoma agentami, którzy handlują jednym dobrem. Każdy pakiet sprzedawanego przez agenta dobra posiada wartość społeczną, zawierającą odpowiednio inwestycje, koszty i spodziewane zyski. Dany agent może sprzedawać pakiet dobra po cenie, która odzwierciedla tę wartość. W pracy, która jest rozwinięciem niektórych wyników otrzymanych wcześniej przez Ferrarę [3], skupiamy swoją uwagę na problemie stabilności ceny negocjowanej przez superzrównoważone koalicje.

Słowa kluczowe: *gry kooperacyjne z transferowalną użytecznością, superzrównoważona koalicja*