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**Struktura zależności dla rozwiązań
ułamkowych równań z szumem
 α -stabilnym**

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The dependence structure of the
solutions of the fractional equations with
 α -stable noise

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Chapter 1

Introduction

1.1 Scope of the paper

The property of *long-range dependence* (or *long memory*) refers to a phenomenon in which the events that are arbitrarily distant still influence each other. Mathematical description of long memory is in terms of the rate of decay of correlations. If the correlations are not absolutely summable, we say that the process exhibits long-range dependence. It requires skill and experience to construct a process with the corresponding correlations decaying to zero slower than exponentially. Most well known stationary processes, such as ARMA models, finite-state Markov chains and Gaussian Ornstein-Uhlenbeck processes lead to exponentially decaying correlations. However, the recent developments in the field of long-range dependence show that the methods of the *fractional calculus* (integrals and derivatives of fractional order) seem to be very promising mathematical tool in constructing processes with long memory.

In the presented thesis, we use the methods of fractional calculus in order to construct stochastic processes with long memory. We concentrate our efforts on generalizing the standard α -stable Ornstein-Uhlenbeck process. However, since the introduced models have α -stable finite-dimensional distributions, the correlation function is not defined. Therefore, we describe the dependence structure of the examined α -stable processes in the language of other measures of dependence appropriate for models with infinite variance, i.e. *covariation* and *correlation cascade*. We find the fundamental relationship between these two measures of dependence and detect the property of long memory in the introduced models.

The paper is organized as follows: In Chapter 2 we introduce the definition of long-range dependence for processes with finite second moment. We discuss the motivations standing behind the introduced definition. We present the classical Gaussian processes exhibiting long memory and introduce the fractional generalizations of the Gaussian Ornstein-Uhlenbeck process.

In Chapter 3 we extend the definition of long-range dependence to the models with infinite variance. We discuss the properties of the measure of dependence

called codifference. We also introduce the recently developed concept of correlation cascade. We show that the correlation cascade is a proper mathematical tool for exploring the dependence structure of infinitely divisible processes. We prove the revised version of the classical Maruyama's mixing theorem for infinitely divisible processes. This result is presented in article [17]. As a consequence, we describe the ergodic properties (ergodicity, weak mixing, mixing) of such processes in the language of correlation cascade. We establish the relationship between both discussed measures of dependence.

In Chapter 4 we introduce four fractional generalizations of the α -stable Ornstein-Uhlenbeck process. We derive precise formulas for the asymptotic behaviour of their codifferences. We verify the property of long memory in the examined models. We define the continuous-time counterpart of FARIMA (fractional autoregressive integrated moving average) time series and prove that it has exactly the same dependence structure as FARIMA. Most of the result of Chapter 4 are presented in articles [16, 18, 19].

In Chapter 5 we use the correlation cascade to examine the dependence structure of the fractional models introduced in Chapter 4. We detect the property of long-range dependence in the language of correlation cascade and show that the results are analogous to the ones for codifference. Using the results from Chapter 3, we verify the ergodic properties of the discussed processes by proving that they are mixing.

The last Chapter summarizes the results of the thesis and presents brief conclusions.

Chapter 2

Long-range dependence in finite variance case

2.1 Foundations

The concept of long-range dependence (or long memory) dates back to a series of papers by Mandelbrot et. al. [20–22] that explained and proposed the appropriate mathematical model for the unusual behaviour of the water levels in the Nile river. Since then this concept has become particularly important in a wide range of applications starting with hydrology, ending with network traffic and finance. The typical way of defining long memory in the time domain is in terms of the rate of decay of the correlation function [2, 7].

Definition 1. A stationary process $\{X(t), t \in \mathbb{R}\}$ with finite second moment is said to have long memory if the following condition holds

$$\sum_{n=0}^{\infty} |Corr(n)| = \infty. \quad (2.1)$$

Here

$$Corr(n) = \frac{E[X(n)X(0)] - E[X(n)]E[X(0)]}{\sqrt{Var[X(n)]}\sqrt{Var[X(0)]}}$$

is the correlation function. Conversely, the process $X(t)$ is said to have *short memory* if the series (2.1) is convergent. Thus, the long-range dependence can be fully characterized by the asymptotic behaviour of the correlation function (or equivalently covariance function).

To understand, why the definition of long-range dependence is based on the lack of summability of correlations, let us consider the following example. Let $\{X(n), n = 0, 1, 2, \dots\}$ be a centered stationary stochastic process with finite variance σ^2 and correlations ρ_n . For the partial sums $S(n) = X(0) + \dots + X(n-1)$ we have

$$Var[S(n)] = \sigma^2 \left(n + 2 \sum_{i=1}^{n-1} (n-i)\rho_i \right).$$

When the correlations ρ_i are summable, the dominated convergence theorem yields

$$\lim_{n \rightarrow \infty} \frac{\text{Var}[S(n)]}{n} = \sigma^2 \left(1 + 2 \sum_{i=1}^{\infty} \rho_i \right).$$

Thus, the variance of the partial sums decays linearly fast. Once the correlations stop being summable, the variance of the partial sums can grow faster than linearly and the actual rate of increase of $\text{Var}[S(n)]$ is related to the rate of decay of correlation. For instance, if

$$\rho_n \sim n^{-d} \text{ as } n \rightarrow \infty,$$

where $0 < d < 1$, then one can verify that

$$\text{Var}[S(n)] \sim \text{const} \cdot n^{2-d} \text{ as } n \rightarrow \infty.$$

As we can see, when the correlations stop being summable, a phase transition in the behaviour of the variance of the partial sums occurs. The rate of increase of $\text{Var}[S(n)]$ depends on the parameter d , which characterizes the asymptotic behaviour of ρ_n . Thus, the lack of summability of correlations causes the phase transition in the asymptotic dependence structure of the process and influences its memory structure.

Another reason for defining long-range dependence in terms of the non-summability of correlations is the existence of a threshold that separates short and long memory for Fractional Gaussian Noise (FGN). To define FGN, we need to recall the definition of the Fractional Brownian Motion (FBM).

Definition. A centered Gaussian stochastic process $\{B_H(t), t \geq 0\}$, $0 < H \leq 1$, with the covariance function $\text{Cov}(B_H(s), B_H(t))$ given by

$$\text{Cov}(B_H(s), B_H(t)) = \frac{1}{2} [t^{2H} + s^{2H} - |t - s|^{2H}] \quad (2.2)$$

is called FBM.

$B_H(t)$ is a stationary-increment, H -self-similar stochastic process. It has found a wide range of applications in modelling various real-life phenomena exhibiting self-similar scaling properties. For $H = 1/2$ the FBM becomes the standard Brownian Motion (or Wiener Process). FBM was first used by Mandelbrot and Van Ness [21] to give a probabilistic model consistent with an unusual behaviour of water levels in the Nile River observed by Hurst [10]. $B_H(t)$ has the following, very useful, integral representation

$$B_H(t) = c_H \cdot \int_{\mathbb{R}} \left[(t-s)_+^{H-1/2} - (-s)_+^{H-1/2} \right] dB(s), \quad (2.3)$$

where $(x)_+ := \max\{x, 0\}$, c_H is the normalizing constant dependent only on H and $B(t)$ is the standard Brownian motion. Since $B_H(t)$ has stationary increments, we can introduce the following stationary sequence

Definition. An increment process $\{b_H(n), n = 0, 1, 2, \dots\}$ defined as

$$b_H(n) = B_H(n+1) - B_H(n) \quad (2.4)$$

is called FGN.

FGN is a stationary and centered Gaussian stochastic process. An immediate conclusion from (2.2) is that for $H \neq 1/2$

$$\rho_n = \text{Corr}(b_H(0), b_H(n)) \sim 2H(2H-1)n^{-2(1-H)}. \quad (2.5)$$

Let us observe that for $H = 1/2$ the FGN is an i.i.d sequence (this follows from the fact that the increments of the Brownian motion are independent and stationary), which implies that the process has no memory. Therefore, the case $H = 1/2$ is considered the threshold that separates short and long memory for the FGN. The correlation function ρ_n of an FGN with $H > 1/2$ decays slower than $1/n$ and in this case $b_H(n)$ is viewed as long-range dependent. Note that for $H > 1/2$ the correlations ρ_n are positive and fulfill condition (2.1). For $H < 1/2$ the correlations ρ_n decay faster than $1/n$ and the process $b_H(n)$ is said to have short memory. Note that in this case the correlations are negative and summable.

The above considerations clearly show that the lack of summability of correlations strongly influences the asymptotic dependence structure of stationary processes. Therefore, the definition of long memory in terms of the rate of decay of correlations is justifiable and well-posed.

2.2 Gaussian fractional Ornstein-Uhlenbeck processes

The classical Gaussian Ornstein-Uhlenbeck (O-U) process $\{Y(t), t \in \mathbb{R}\}$ is one of the most well known and explored stationary stochastic processes. It can be equivalently defined in the three following ways:

- (i) as the centered Gaussian process with the correlation function given by

$$\text{Corr}[Y(s), Y(t)] = \exp\{-\lambda|s-t|\}, \lambda > 0,$$

- (ii) as the Lamperti transformation [4] of the classical Brownian motion $B(t)$, i.e.

$$Y(t) = e^{-\lambda t} B(e^{2\lambda t}),$$

Recall that the Lamperti transformation provides one-to-one correspondence between self-similar and stationary processes.

- (iii) as the stationary solution of the Langevin equation

$$\frac{dY(t)}{dt} + \lambda Y(t) = b(t), \quad (2.6)$$

where $b(t)$ is the Gaussian white noise, heuristically $b(t) = dB(t)/dt$.

It is worth mentioning that the O-U process has the following, very useful, moving-average integral representation

$$Y(t) = \int_{-\infty}^t e^{-\lambda(t-s)} dB(s). \quad (2.7)$$

Since the correlation function of the O-U process decays exponentially, condition (2.1) is not fulfilled. Thus, $Y(t)$ is a short memory process. This is not surprising, since $Y(t)$ is Markovian. The exponential decay of correlation is typical for most well known stationary processes. It is enough to say that all ARMA models, GARCH time series and finite-state Markov chains lead to exponentially decaying correlations. A process with correlations decaying slower than exponentially is, therefore, unusual. It requires a lot of skill to construct a stationary process with the corresponding correlations decaying to zero slower than exponentially. Therefore, it is of great interest to develop an approach, which will let us construct processes with non-summable correlations. A prominent example of such process is the FGN, defined through the increments of the FBM (see (2.4)). Let us note that the FBM can be viewed as the fractional generalization of the standard Brownian motion. Additionally, such fractional generalization results in a transition from a process with no memory (i.i.d sequence of increments of the Brownian motion) to a process with long memory (FGN). Therefore, it promises well to check if a similar transition from short to long memory process occurs, when considering fractional generalizations of the O-U process.

The first fractional generalization of the O-U process $Y(t)$ is obtained in the following way. Since $Y(t)$ can be defined as the Lamperti transformation of the Brownian motion (see definition (ii)), the fractional O-U process of the first kind $\{Y_1(t), t \in \mathbb{R}\}$ is introduced as the Lamperti transformation of the FBM, i.e.

$$Y_1(t) = e^{-tH} B_H(e^t), \quad 0 < H \leq 1. \quad (2.8)$$

$Y_1(t)$ is a stationary, centered Gaussian process. For $H = 1/2$ it becomes the standard O-U process. The dependence structure of $Y_1(t)$ was studied by Cheridito et.al [6]. The authors showed that the covariance function of $Y_1(t)$ satisfies

$$\text{Cov}(Y_1(0), Y_1(t)) \sim c_H \cdot e^{-t(H \wedge (1-H))}$$

as $t \rightarrow \infty$. Here c_H is the appropriate constant and $(x \wedge y) := \min\{x; y\}$. Therefore, the correlations of $Y_1(t)$ also decay exponentially, which implies that the first considered fractional Ornstein-Uhlenbeck process does not have long-range dependence property.

One can also consider the finite-memory part of $B_H(t)$ given by the following Riemann-Liouville fractional integral (see [28])

$$\tilde{B}_H(t) = \Gamma(H + 1/2)^{-1} \int_0^t (t-s)^{H-1/2} dB(s), \quad t > 0, \quad H > 0. \quad (2.9)$$

Here $\Gamma(\cdot)$ is the gamma function. The process $\tilde{B}_H(t)$ is called the finite-memory FBM [14, 23]. It is clearly H -self-similar, but unlike $B_H(t)$, it does not have stationary increments. The Lamperti transformation of $\tilde{B}_H(t)$ gives the second fractional generalization of the O-U process

$$Y_2(t) = e^{-tH} \tilde{B}_H(e^t), \quad t \in \mathbb{R}. \quad (2.10)$$

However, as shown in [14], the correlations of $Y_2(t)$ also decay exponentially, thus the process has short memory.

The next generalization $\{Y_3(t), t \in \mathbb{R}\}$ is obtained by 'fractionalizing' the Langevin equation (2.6) in the following manner

$$\left(\frac{d}{dt} + \lambda\right)^\kappa Y_3(t) = b(t), \quad \kappa > 0, \lambda > 0. \quad (2.11)$$

Here the operator $\left(\frac{d}{dt} + \lambda\right)^\kappa$ is the so-called modified Bessel derivative (see Sec.4.3 and [28] for more details). Note that for $\kappa = 1$ the above equation becomes the standard Langevin equation and its stationary solution is the O-U process. To solve equation (2.11), we apply the standard Fourier transform techniques. Hence, we obtain

$$Y_3(t) = \Gamma(\kappa)^{-1} \int_{-\infty}^t (t-s)^{\kappa-1} e^{-\lambda(t-s)} dB(s). \quad (2.12)$$

For $\kappa > 1/2$ the stochastic integral is well defined in the sense of convergence in probability and the process $Y_3(t)$ is properly defined. The covariance function of $Y_3(t)$ satisfies

$$Cov(Y_3(0), Y_3(t)) = \frac{1}{\Gamma^2(\kappa)} \int_0^\infty s^{\kappa-1} e^{-\lambda s} (s+t)^{\kappa-1} e^{-\lambda(t+s)} ds.$$

Thus, from the dominated convergence theorem, we immediately obtain that the covariance function decays exponentially. As a consequence, we get that $Y_3(t)$ is also a short memory process.

The last generalization $\{Y_4(t), t \in \mathbb{R}\}$ is obtained by replacing the Gaussian white noise $b(t)$ in (2.6) with the fractional noise $b_H(t)$. Formally, $b_H(t) = \frac{dB_H(t)}{dt}$. Thus, the process $Y_4(t)$ is defined as the stationary solution of the following fractional Langevin equation

$$\frac{dY_4(t)}{dt} + \lambda Y_4(t) = b_H(t). \quad (2.13)$$

The above equation can be rewritten in the equivalent, perhaps more convenient, form

$$dY_4(t) = -\lambda Y_4(t)dt + dB_H(t).$$

Obviously, for $H = 1/2$ it becomes the standard Langevin equation. As shown in [6], the unique stationary solution of (2.13) has the form

$$Y_4(t) = \int_{-\infty}^t e^{-\lambda(t-s)} dB_H(s), \quad (2.14)$$

where the stochastic integral is understood as the Riemann-Stieltjes integral. In [6], the authors show that the covariance function of $Y_4(t)$ satisfies

$$\text{Cov}(Y_4(0), Y_4(t)) \sim d_H t^{-2(1-H)}$$

as $t \rightarrow \infty$. Here d_H is the appropriate non-zero constant. Therefore, for $H > 1/2$ the correlations are not summable and the process has long memory. Note that the asymptotic behaviour of the covariance function of $Y_4(t)$ is analogous to the behaviour of the correlations of the FGN. We can, therefore, conclude that the long memory of the increments of $B_H(t)$ transfers to the solution of the fractional Langevin equation (2.13).

As we can see, only the last fractional generalization of the O-U process resulted in a process with long memory, which confirms the fact that the faster than exponential decay of correlations is 'unusual'. Let us note that the presented considerations were limited only to the Gaussian distributions. In what follows, we extend our investigations concerning the notion of long-range dependence to the more general case of α -stable distributions.

Chapter 3

Long-range dependence in infinite variance case

3.1 Codifference

Historically, long-range dependence is measured in terms of summability of correlations. This approach was introduced and discussed in details in the previous chapter. However, the situation becomes more complicated, while considering processes with infinite variance, in particular, processes with α -stable marginal distributions, $0 < \alpha < 2$ (see [11, 29]). In α -stable case, the correlations can no longer be calculated and the definition of long memory has to be reformulated. Since there are no correlations to look at, one has to look at the substitute measure of dependence. The first thought that comes to mind, while searching for a measure of dependence for α -stable distributions, is about *codifference*. It is defined in the following way:

Definition ([29]). The codifference $\tau_{X,Y}$ of two jointly α -stable random variables X and Y equals

$$\tau_{X,Y} = \ln Ee^{i(X-Y)} - \ln Ee^{iX} - \ln Ee^{-iY}. \quad (3.1)$$

The codifference shares the following important properties:

- It is always well-defined, since the definition of $\tau_{X,Y}$ is based on the characteristic functions of α -stable random variables X and Y .
- When $\alpha = 2$, the codifference reduces to the covariance $Cov(X, Y)$.
- If the random variables X and Y are symmetric, then $\tau_{X,Y} = \tau_{Y,X}$.
- If X and Y are independent, then $\tau_{X,Y} = 0$. Conversely, if $\tau_{X,Y} = 0$ and $0 < \alpha < 2$, then X and Y are independent. When $1 \leq \alpha < 2$, $\tau_{X,Y} = 0$ does not imply that X and Y are independent (see [29]).

- Let (X, Y) and (X', Y') be two symmetric α -stable random vectors and let all random variables X, X', Y, Y' have the same scale parameters. Then the following inequality holds [29]: If

$$\tau_{X,Y} \leq \tau_{X',Y'},$$

then for every $c > 0$ we have

$$P\{|X - Y| > c\} \geq P\{|X' - Y'| > c\}.$$

The above inequality has the following interpretation: the random variables X' and Y' are less likely to differ than X and Y , thus they are more dependent. Therefore, the larger τ , the 'greater' the dependence.

The above properties confirm that the codifference is the appropriate mathematical tool for measuring the dependence between the α -stable random variables. In what follows, we will be mostly interested in investigating the asymptotic behaviour of the function

$$\tau(t) := \tau_{Y(0), Y(t)}, \quad (3.2)$$

where $\{Y(t), t \in \mathbb{R}\}$ is a stationary α -stable process. It is worth noticing that $\tau(t)$ tends to zero as $t \rightarrow \infty$, if the process $Y(t)$ is a symmetric α -stable moving average, i.e., a process of the form

$$Y(t) = \int_{\mathbb{R}} f(t-s) M(ds),$$

where M is a symmetric α -stable random measure with Lebesgue control measure, while f is measurable and α -integrable. Surprisingly, $\tau(t)$ carries enough information to detect the chaotic properties (ergodicity, mixing) for the class of stationary infinitely divisible processes (see next section and [26, 27] for the details).

Now, as a straightforward extension of (2.1), we introduce the following definition of long memory in the α -stable case

Definition 2. A stationary α -stable process $\{Y(t), t \in \mathbb{R}\}$ is said to have long memory if the following condition holds

$$\sum_{n=0}^{\infty} |\tau(n)| = \infty. \quad (3.3)$$

The above definition indicates that long-range dependence in the α -stable case will be measured in terms of the rate of decay of the codifference. Obviously, when $\alpha = 2$ the definitions (2.1) and (3.3) are equivalent.

In the literature one can also find the quantity parametrized by $\theta_1, \theta_2 \in \mathbb{R}$, which is closely related to the codifference $\tau(t)$. It is defined as [29]

$$\begin{aligned} I(\theta_1; \theta_2; t) &= -\ln E[\exp\{i(\theta_1 Y(t) + \theta_2 Y(0))\}] \\ &+ \ln E[\exp\{i\theta_1 Y(t)\}] + \ln E[\exp\{i\theta_2 Y(0)\}]. \end{aligned} \quad (3.4)$$

We call it the *generalized codifference*, since $\tau(t) = -I(1; -1; t)$. The presence of the parameters θ_1 and θ_2 in the definition of $I(\cdot)$ has the following advantage: consider two stationary α -stable stochastic processes Y and Y' . In order to show that the two processes are different, we examine the asymptotic behaviour of the corresponding measures of dependence $I_Y(\theta_1; \theta_2; t)$ and $I_{Y'}(\theta_1; \theta_2; t)$. If the measures are not asymptotically equivalent at least for one specific choice of θ_1 and θ_2 , then the processes Y and Y' must be different.

The generalized codifference $I(\cdot)$ (to be precise, the function asymptotically equivalent to $I(\cdot)$) was used in [13] for distinguishing between the asymptotic structures of the moving average, sub-Gaussian and real harmonizable processes. It was also employed in [1] to explore the dependence structure of the fractional α -stable noise.

Recall that in the Gaussian case the classical example of the long-memory process was the FGN (2.4), defined as the increment process of the FBM. The extension of the FBM to the α -stable case is called the *fractional α -stable motion* and defined as:

Definition Let $0 < \alpha \leq 2$, $0 < H < 1$, $H \neq 1/\alpha$ and $a, b \in \mathbb{R}$, $|a| + |b| > 0$. Then the process

$$\begin{aligned} L_{\alpha, H}(t) = & \int_{-\infty}^{\infty} \left(a \left[(t-s)_+^{H-1/\alpha} - (-s)_+^{H-1/\alpha} \right] \right. \\ & \left. + b \left[(t-s)_-^{H-1/\alpha} - (-s)_-^{H-1/\alpha} \right] \right) L_{\alpha}(ds), \quad t \in \mathbb{R}, \end{aligned} \quad (3.5)$$

is called **fractional α -stable motion**.

Here $x_+ = \max\{x, 0\}$, $x_- = \max\{-x, 0\}$ and $L_{\alpha}(s)$ is the standard symmetric α -stable random measure on \mathbb{R} with control measure as Lebesgue measure, [11, 29]. $L_{\alpha, H}(t)$ is a H -self-similar, stationary-increment process [32]. For $\alpha = 2$ it reduces to the fractional Brownian motion.

Now, the *fractional α -stable noise* $l_{\alpha, H}$ is a stationary sequence defined as the increment process of $L_{\alpha, H}$, i.e.

$$l_{\alpha, H}(n) = L_{\alpha, H}(n+1) - L_{\alpha, H}(n), \quad n = 0, 1, \dots \quad (3.6)$$

For $\alpha = 2$ the process $l_{\alpha, H}$ reduces to the FGN. The following result was proved in [1]

Theorem. ([1]) *The generalized codifference of $l_{\alpha, H}$ satisfies*

(i) *If either $0 < \alpha \leq 1$, $0 < H < 1$ or $1 < \alpha < 2$, $1 - \frac{1}{\alpha(\alpha-1)} < H < 1$, $H \neq 1/\alpha$ then*

$$I(\theta_1; \theta_2, n) \sim B(\theta_1; \theta_2) n^{\alpha H - \alpha}$$

as $n \rightarrow \infty$.

(ii) *If $1 < \alpha < 2$, $0 < H < 1 - \frac{1}{\alpha(\alpha-1)}$ then*

$$I(\theta_1; \theta_2, n) \sim C(\theta_1; \theta_2) n^{H-1/\alpha-1}$$

as $n \rightarrow \infty$.

Here $B(\theta_1; \theta_2)$ and $C(\theta_1; \theta_2)$ are the appropriate non-zero constants.

As a consequence, we get

Corollary 1. *For $H > 1/\alpha$ the process $l_{\alpha, H}$ has long memory in the sense of (3.3).*

Note that for $\alpha = 2$ the above result reduces to the one obtained for the FGN (see (2.5)).

In the next chapter we investigate the dependence structure of the fractional α -stable O-U processes and compare the results to the ones known from the Gaussian case. But first, let us introduce the recently developed concept of *correlation cascade* – an alternative measure of dependence for α -stable processes..

3.2 Correlation Cascade

3.2.1 Definition and basic properties

Let us consider an infinitely divisible (i.d.), [30], stochastic process $\{Y(t), t \in \mathbb{R}\}$ with the following integral representation

$$Y(t) = \int_X K(t, x) N(dx). \quad (3.7)$$

Here N is an independently scattered i.d. random measure on some measurable space X with a control measure m , such that for every m -finite set $A \subseteq X$ we have (Lévy-Khinchin formula)

$$E \exp[izN(A)] = \exp \left[m(A) \left\{ iz\mu - \frac{1}{2}\sigma^2 z^2 + \int_{\mathbb{R}} (e^{izx} - 1 - izx\mathbf{1}(|x| < 1)) Q(dx) \right\} \right].$$

The random measure N is fully determined by the control measure m , the Lévy measure Q , the variance of the Gaussian part σ^2 and the drift parameter $\mu \in \mathbb{R}$. Additionally, the kernel $K(t, x)$ is assumed to take only nonnegative values.

Since, in general, the second moment and thus the correlation function for the process $Y(t)$ may be infinite, the key problem is, how to describe mathematically the underlying dependence structure of $Y(t)$. In the recent paper by Eliazar and Klafter [8], authors introduce a new concept of *Correlation Cascade*, which is a promising tool for exploiting the properties of the Poissonian part of $Y(t)$ and the dependence structure of this stochastic process. They proceed in the following way: First, let us define the Poissonian tail-rate function Λ of the Lévy measure Q as

$$\Lambda(l) = \int_{|x|>l} Q(dx), \quad l > 0, \quad (3.8)$$

Now, we introduce

Definition 3. For $t_1, \dots, t_n \in \mathbb{R}$ and $l > 0$ the function

$$C_l(t_1, \dots, t_n) = \int_X \Lambda\left(\frac{l}{\min\{K(t_1, x), \dots, K(t_n, x)\}}\right) m(dx), \quad (3.9)$$

is called the **Correlation Cascade**.

As shown in [8], with the help of the function $C_l(t_1, \dots, t_n)$ one can determine the distributional properties of the Poissonian part of $Y(t)$ and describe the correlation-like structure of the process. Recall that the i.d. random measure N in (3.7) admits the following stochastic representation (Lévy-Ito formula)

$$\begin{aligned} N(B) &= \mu \cdot m(B) + N_G(B) + \int_B \int_{|y|>1} y N_P(dx \times dy) \\ &+ \int_B \int_{|y|\leq 1} y (N_P(dx \times dy) - m_P(dx \times dy)), \end{aligned} \quad (3.10)$$

where $N_G(B)$ is a Gaussian random variable with mean zero and standard deviation equal to $\sigma \cdot m(B)$, while N_P is the Poisson point process with the control measure $m_P = m \times Q$. Now, for $l > 0$, let us introduce the random variable

$$\Pi_l(t) = \int_X \int_{|y|>0} \mathbf{1}_{\{|yK(t,x)|>l\}} N_P(dx \times dy). \quad (3.11)$$

$\Pi_l(t)$ has the following interpretation: it is the number of elements of the set $\{yK(t, x) : (x, y) \text{ is the atom of the Poisson point process } N_P\}$ whose absolute value is greater than the level l . It is of great importance to know the relationship between the random variables $\Pi_l(t)$ and the correlation cascade $C_l(\cdot)$. As shown in [8], the following formulas, which explain the meaning of $C_l(\cdot)$, hold true

$$\begin{aligned} E[\Pi_l(t)] &= C_l(t), \\ Cov[\Pi_l(t_1), \Pi_l(t_2)] &= C_l(t_1, t_2), \\ Corr[\Pi_l(t_1), \Pi_l(t_2)] &= \frac{C_l(t_1, t_2)}{\sqrt{C_l(t_1)C_l(t_2)}} \end{aligned} \quad (3.12)$$

In what follows, we establish the relationship between $C_l(t_1, \dots, t_n)$ and the corresponding Lévy measure ν_{t_1, \dots, t_n} of the i.d. random vector $(Y(t_1), \dots, Y(t_n))$. The result will allow us to give a new meaning to the function $C_l(t_1, \dots, t_n)$ and to recognize it as an appropriate instrument for characterizing the dependence structure of $Y(t)$. We prove the following result

Proposition 1. Let $Y(t)$ be of the form (3.7) and let ν_{t_1, \dots, t_n} be the Lévy measure of the i.d. random vector $(Y(t_1), \dots, Y(t_n))$. Then, the corresponding function $C_l(\cdot)$ given in (3.9) satisfies

$$C_l(t_1, \dots, t_n) = \nu_{t_1, \dots, t_n}(\{(x_1, \dots, x_n) : \min\{|x_1|, \dots, |x_n|\} > l\}). \quad (3.13)$$

PROOF. Using the relationship between the measures Q and ν_{t_1, \dots, t_n} (see [25] for details), we obtain

$$\begin{aligned}
C_l(t_1, \dots, t_n) &= \int_X \Lambda\left(\frac{l}{\min\{K(t_1, x), \dots, K(t_n, x)\}}\right) m(dx) = \\
&= \int_X \int_{\mathbb{R}} \mathbf{1}\left(|y| > \frac{l}{\min\{K(t_1, x), \dots, K(t_n, x)\}}\right) Q(dy) m(dx) = \\
&= \int_X \int_{\mathbb{R}} \mathbf{1}(\min\{|yK(t_1, x)|, \dots, |yK(t_n, x)|\} > l) Q(dy) m(dx) = \\
&= \int_{\mathbb{R}^n} \mathbf{1}(\min\{|x_1|, \dots, |x_n|\} > l) \nu_{t_1, \dots, t_n}(dx_1, \dots, dx_n) = \\
&= \nu_{t_1, \dots, t_n}(\{(x_1, \dots, x_n) : \min\{|x_1|, \dots, |x_n|\} > l\}). \blacksquare
\end{aligned}$$

Since, for an i.d. vector $\underline{Y} = (Y(t_1), \dots, Y(t_n))$, the independence of the coordinates $Y(t_1), \dots, Y(t_n)$ is equivalent to the fact that the Lévy measure of \underline{Y} is concentrated on the axes, the above result gives a new meaning to the function C_l . Namely, $C_l(t_1, \dots, t_n)$ indicates, how much mass of the measure ν_{t_1, \dots, t_n} is concentrated beyond the axes and their l -surrounding (here by l -surrounding we mean the set $\{(x_1, \dots, x_n) : \min\{|x_1|, \dots, |x_n|\} \leq l\}$). In other words, **the function $C_l(\mathbf{t}_1, \dots, \mathbf{t}_n)$ tells us, how dependent the coordinates of the vector $(Y(\mathbf{t}_1), \dots, Y(\mathbf{t}_n))$ are.** Therefore, $C_l(t_1, \dots, t_n)$ can be considered an appropriate measure of dependence for the Poissonian part of the i.d. process $Y(t)$. In particular, the function $C_l(t_1, t_2)$ can serve as an analogue of the covariance and the function

$$r_l(t_1, t_2) := \frac{C_l(t_1, t_2)}{\sqrt{C_l(t_1)C_l(t_2)}} \quad (3.14)$$

can play the role of the correlation coefficient.

Let us now consider the case, when the random measure N is α -stable. In such setting, the Lévy measure Q in the Lévy-Khinchin representation has the form

$$Q(dx) = \frac{c_1}{x^{1+\alpha}} \mathbf{1}_{(0, \infty)}(x) dx + \frac{c_2}{|x|^{1+\alpha}} \mathbf{1}_{(-\infty, 0)}(x) dx,$$

where c_1 and c_2 are the appropriate constants. Consequently, the tail function is given by

$$\Lambda(l) = C \cdot l^{-\alpha}$$

and for the correlation cascade we get

$$C_l(t_1, \dots, t_n) = C \cdot l^{-\alpha} \int_X \min\{K(t_1, x), \dots, K(t_n, x)\}^\alpha m(dx),$$

where C is the appropriate constant. From the last formula we get that the correlation-like function $r_l(t_1, t_2)$ given by (3.14) does not depend on the parameter l . We have

$$r(t_1, t_2) := r_l(t_1, t_2) = \frac{\int_X \min\{K(t_1, x), K(t_2, x)\}^\alpha m(dx)}{\sqrt{\int_X K(t_1, x)^\alpha m(dx) \int_X K(t_2, x)^\alpha m(dx)}}. \quad (3.15)$$

The function $r(t_1, t_2)$ plays the role of the correlation in the α -stable case and measures the dependence between the random variables $Y(t_1)$ and $Y(t_2)$. Now, let us consider a stationary α -stable stochastic process $\{Y(t), t \in \mathbb{R}\}$. Since $Y(t)$ is stationary, $r(\tau, \tau + t)$ does not depend on τ . Therefore, the function

$$r(t) := r(\tau, \tau + t) = \frac{\int_X \min\{K(t, x), K(0, x)\}^\alpha m(dx)}{\int_X K(0, x)^\alpha m(dx)} \quad (3.16)$$

can be considered a correlation-like measure of dependence for stationary α -stable process $Y(t)$. The immediate consequence is the following, alternative to (3.3), definition of long memory in α -stable case

Definition 4. *A stationary α -stable process $\{Y(t), t \in \mathbb{R}\}$ is said to have long memory in terms of the correlation cascade if the following condition holds*

$$\sum_{n=0}^{\infty} |r(n)| = \infty, \quad (3.17)$$

where $r(\cdot)$ is given by (3.16).

Note that

$$r(t) = \frac{C_l(0, t)}{C \cdot l^{-\alpha} \int_X K(0, x)^\alpha m(dx)},$$

thus, in order to verify the long memory property of $Y(t)$, it is enough to examine the asymptotic behaviour of $C_l(0, t)$.

In the previous section, we discussed the dependence structure and the property of long memory for the fractional α -stable noise $l_{\alpha, H}$ (3.6) in terms of the codifference. It is of great interest to verify if the process $l_{\alpha, H}$ displays long-range dependence also in the sense of (3.17). As shown in [8], the correlation-like function $r(t)$ corresponding to $l_{\alpha, H}$ satisfies

$$r(t) \sim t^{\alpha H - \alpha}$$

as $t \rightarrow \infty$. As a consequence, we obtain

Corollary 2. *For $H > 1/\alpha$ the process $l_{\alpha, H}(t)$ has long memory in the sense of (3.17).*

Note that this result is analogous to the one for codifference (compare with Corollary 1). However, the question arises, if the similar analogy can be observed for the fractional O-U processes. This issue will be discussed in details in chapters 4 and 5.

Let us emphasize that the concept of long memory in the non-Gaussian world is still not well-formulated and is a subject of many extensive research. Therefore, the introduced definitions of long-range dependence and the obtained results should be viewed as one possible approach to long memory for processes with infinite variance.

In the next section we describe the ergodic properties of i.d. processes in the language of correlation cascade $C_l(0, t)$. As a consequence, we obtain the relationship between $C_l(0, t)$ and the codifference $\tau(t)$.

3.2.2 Ergodicity, weak mixing and mixing

In this section we prove the revised version of the classical Maruyama's mixing theorem [24]. As a consequence, we describe the ergodic properties (ergodicity, weak mixing, mixing) of i.d. processes in the language of correlation cascade. We use the obtained results to establish the relationship between both previously introduced measures of dependence – codifference and correlation cascade.

Be begin with recalling some basic facts from ergodic theory. Let $\{Y(t), t \in \mathbb{R}\}$ be a stationary, i.d. stochastic process defined on the canonical space $(\mathbb{R}^{\mathbb{R}}, \mathcal{F}, P)$. The process $Y(t)$ is said to be **ergodic** if

$$\frac{1}{T} \int_0^T P(A \cap S^t B) dt \longrightarrow P(A)P(B) \text{ as } T \rightarrow \infty, \quad (3.18)$$

weakly mixing if

$$\frac{1}{T} \int_0^T |P(A \cap S^t B) - P(A)P(B)| dt \longrightarrow 0 \text{ as } T \rightarrow \infty, \quad (3.19)$$

mixing if

$$P(A \cap S^t B) \longrightarrow P(A)P(B) \text{ as } t \rightarrow \infty, \quad (3.20)$$

for every $A, B \in \mathcal{F}$, where (S^t) is a group of shift transformations on $\mathbb{R}^{\mathbb{R}}$.

The description of the mixing property for stationary i.d. processes in terms of their Lévy characteristics dates back to the fundamental paper by Maruyama [24]. He proved the following result

Theorem. ([24]) *An i.d. stationary process $Y(t)$ is mixing if and only if*
(C1) the covariance function $\text{Cov}(t)$ of its Gaussian part converges to 0 as $t \rightarrow \infty$,
(C2) $\lim_{t \rightarrow \infty} \nu_{0t}(|xy| > \delta) = 0$ for every $\delta > 0$, and
(C3) $\lim_{t \rightarrow \infty} \int_{0 < x^2 + y^2 \leq 1} xy \nu_{0t}(dx, dy) = 0$,
where ν_{0t} is the Lévy measure of $(Y(0), Y(t))$.

The above result was crucial for further scientific research on the subject of ergodic properties of stochastic processes, and has been extensively exploited by many authors (see, eg. [9, 11, 26]). In what follows, we show that condition (C2) implies (C3), and therefore the necessary and sufficient conditions for an i.d. process to be mixing can be reduced only to (C1) and (C2).

Lemma 1. *Assume that $\lim_{t \rightarrow \infty} \nu_{0t}(|xy| > \delta) = 0$ for every $\delta > 0$. Then, we get*

$$\lim_{t \rightarrow \infty} \int_{0 < x^2 + y^2 \leq 1} xy \nu_{0t}(dx, dy) = 0.$$

PROOF. First, let us notice that the assumption in the lemma implies that

$$\lim_{t \rightarrow \infty} \nu_{0t}(|x| \wedge |y| > l) = 0 \quad \text{for every } l > 0, \quad (3.21)$$

where $a \wedge b = \min\{a, b\}$. Indeed, putting $\delta = l^2$, we get

$$\nu_{0t}(|x| \wedge |y| > l) \leq \nu_{0t}(|xy| > \delta) \longrightarrow 0$$

as $t \rightarrow \infty$.

Now, fix $\epsilon > 0$, put $B_\delta = \{x^2 + y^2 \leq \delta^2\}$ and $R_\delta = \{\delta^2 < x^2 + y^2 \leq 1\}$. Then, we obtain

$$\int_{0 < x^2 + y^2 \leq 1} |xy| \nu_{0t}(dx, dy) = \int_{B_\delta} |xy| \nu_{0t}(dx, dy) + \int_{R_\delta} |xy| \nu_{0t}(dx, dy) =: I_1 + I_2.$$

We will estimate both terms I_1 and I_2 separately.

Taking advantage of stationarity of ν_{0t} , we get for the first term

$$\begin{aligned} I_1 &\leq \frac{1}{2} \int_{B_\delta} x^2 \nu_{0t}(dx, dy) + \frac{1}{2} \int_{B_\delta} y^2 \nu_{0t}(dx, dy) \leq \\ &\leq \frac{1}{2} \int_{\{x \leq \delta^2\}} x^2 \nu_{0t}(dx, dy) + \frac{1}{2} \int_{\{y^2 \leq \delta^2\}} y^2 \nu_{0t}(dx, dy) = \int_{|x| \leq \delta} x^2 \nu_0(dx). \end{aligned}$$

Thus, for some appropriately small δ_0 we have

$$I_1 = \int_{B_{\delta_0}} |xy| \nu_{0t}(dx, dy) \leq \epsilon/2. \quad (3.22)$$

For the next term, put $l_0 = \min\{\frac{\delta_0}{2}, \frac{\epsilon}{8q}\}$, with $q = \nu_0(|x| > \frac{\delta_0}{2}) < \infty$. Then, for $C = R_{\delta_0} \cap \{|x| \wedge |y| > l_0\}$ we obtain

$$\begin{aligned} I_2 &= \int_C |xy| \nu_{0t}(dx, dy) + \int_{R_{\delta_0} \setminus C} |xy| \nu_{0t}(dx, dy) \leq \nu_{0t}(C) + \int_{R_{\delta_0} \setminus C} \frac{\epsilon}{8q} \nu_{0t}(dx, dy) \leq \\ &\leq \nu_{0t}(|x| \wedge |y| > l_0) + \frac{\epsilon}{8q} \nu_{0t}(R_{\delta_0} \setminus C) \leq \\ &\leq \nu_{0t}(|x| \wedge |y| > l_0) + \frac{\epsilon}{8q} \nu_{0t} \left(\left\{ |x| > \frac{\delta_0}{2} \right\} \cup \left\{ |y| > \frac{\delta_0}{2} \right\} \right) \leq \\ &\leq \nu_{0t}(|x| \wedge |y| > l_0) + \frac{\epsilon}{8q} \nu_{0t} \left(|x| > \frac{\delta_0}{2} \right) + \frac{\epsilon}{8q} \nu_{0t} \left(|y| > \frac{\delta_0}{2} \right) = \\ &= \nu_{0t}(|x| \wedge |y| > l_0) + \frac{\epsilon}{4q} \nu_0 \left(|x| > \frac{\delta_0}{2} \right) = \nu_{0t}(|x| \wedge |y| > l_0) + \frac{\epsilon}{4}. \end{aligned}$$

Using (3.21), for large enough t we have $\nu_{0t}(|x| \wedge |y| > l_0) < \frac{\epsilon}{4}$, and therefore

$$I_2 = \int_{R_{\delta_0}} |xy| \nu_{0t}(dx, dy) < \frac{\epsilon}{2}. \quad (3.23)$$

Finally, combining (4.13) and (3.23), and letting $\epsilon \searrow 0$, we obtain the desired result. \blacksquare

The above result allows us to formulate the following revised version of Maruyama's mixing theorem

Theorem 1. *An i.d. stationary process $Y(t)$ is mixing if and only if the following two conditions hold*

(C1) *the covariance function $\text{Cov}(t)$ of its Gaussian part converges to 0 as $t \rightarrow \infty$, and*

(C2) *$\lim_{t \rightarrow \infty} \nu_{0t}(|xy| > \delta) = 0$ for every $\delta > 0$, where ν_{0t} is the Lévy measure of $(Y(0), Y(t))$.*

PROOF. Necessity follows directly from the Maruyama's theorem. For sufficiency, let us notice, that from Lemma 1 we see that condition (C2) implies (C3). Thus, the process must be mixing. \blacksquare

Remark. Condition (C2) says that the Lévy measure ν_{0t} is asymptotically concentrated on the axes, which for an i.d distribution is equivalent to the asymptotic independence of the Poissonian parts of $Y(0)$ and $Y(t)$. Therefore, conditions (C1) and (C2) yield the asymptotic independence of $Y(0)$ and $Y(t)$, which, in view of definition (3.20), is the natural interpretation of mixing property.

To express the mixing property in the language of the previously introduced (3.9) correlation cascade $C_t(\cdot)$, we prove the following lemma

Lemma 2. *Let $Y(t)$ be an i.d. process and let ν_{0t} be the corresponding Lévy measure of $(Y(0), Y(t))$. Then, the following two conditions are equivalent*

(i) *$\lim_{t \rightarrow \infty} \nu_{0t}(|xy| > \delta) = 0$ for every $\delta > 0$,*

(ii) *$\lim_{t \rightarrow \infty} \nu_{0t}(\min\{|x|, |y|\} > \delta) = 0$ for every $\delta > 0$.*

PROOF. (i) \Rightarrow (ii)

We have

$$\nu_{0t}(\min\{|x|, |y|\} > \delta) \leq \nu_{0t}(|xy| > \delta^2) \longrightarrow 0$$

as $t \rightarrow \infty$.

(ii) \Rightarrow (i)

Fix $\delta > 0$ and $\epsilon > 0$. Denote by ν_0 the Lévy measure of $Y(0)$. Then, there exist $n \in \mathbb{N}$, such that

$$\nu_0(|x| > n) < \frac{\epsilon}{4}.$$

Taking advantage of stationarity of $Y(t)$ we get

$$\begin{aligned} \nu_{0t}(|xy| > \delta) &\leq \nu_{0t}(\min\{|x|, |y|\} > \delta/n) + \nu_{0t}(|x| > n \vee |y| > n) \leq \\ &\leq \nu_{0t}(\min\{|x|, |y|\} > \delta/n) + \nu_0(|x| > n) + Q_t(|y| > n) = \\ &= \nu_{0t}(\min\{|x|, |y|\} > \delta/n) + 2\nu_0(|x| > n) \leq \epsilon/2 + \epsilon/2 = \epsilon \end{aligned}$$

for appropriately large t . Thus, we obtain $\nu_{0t}(|xy| > \delta) \rightarrow 0$ as $t \rightarrow \infty$. ■

As a consequence, we have the following result

Corollary 3. *An i.d. stationary process $Y(t)$ is mixing if and only if the following two conditions hold*

(C1) *the covariance function $Cov(t)$ of its Gaussian part converges to 0 as $t \rightarrow \infty$,*
(C2) *$\lim_{t \rightarrow \infty} \nu_{0t}(\min\{|x|, |y|\} > \delta) = 0$ for every $\delta > 0$, where ν_{0t} is the Lévy measure of $(Y(0), Y(t))$.*

PROOF. Combination of Theorem 1 and Lemma 2 yields the desired result. ■

In what follows, we describe the ergodic properties for the i.d stochastic processes $Y(t)$ of the form (3.7) in the language of the function $C_l(\cdot)$. From now on to the end of the paper we assume for simplicity that the process $Y(t)$ has no Gaussian part.

Let us prove the following theorem

Theorem 2. *Let $Y(t)$ be a stationary i.d. process of the form (3.7). Then $Y(t)$ is mixing iff the corresponding function C_l satisfies*

$$\lim_{t \rightarrow \infty} C_l(0, t) = 0$$

for every $l > 0$.

PROOF. From Proposition 1 we have that

$$C_l(0, t) = \nu_{0t}(\min\{|x|, |y|\} > l).$$

Since the Gaussian part of $Y(t)$ is equal to zero, so is its covariance function. Thus, from Corollary 3 we obtain that $Y(t)$ is mixing iff $\lim_{t \rightarrow \infty} C_l(0, t) = 0$ for every $l > 0$. ■

Example. Let us consider the α -stable moving-average process

$$Y(t) = \int_{-\infty}^t f(t-x) L_\alpha(dx).$$

Here f is assumed to be nonnegative, monotonically decreasing function and $L_\alpha(x)$ is the standard symmetric α -stable random measure on \mathbb{R} with control measure m . In this case the function C_l has the form

$$C_l(0, t) = \text{const} \cdot l^{-\alpha} \int_t^\infty |f(y)|^\alpha m(dy). \quad (3.24)$$

Since f must be α -integrable with respect to the measure m , we get that $\lim_{t \rightarrow \infty} C_l(0, t) = 0$ for every $l > 0$. It implies that **every α -stable moving average is mixing**.

Recall that the function

$$\tau(t) = \log Ee^{i(Y(t)-Y(0))} - \log Ee^{iY(t)} - \log Ee^{-iY(0)},$$

called codifference (3.2), is an alternative measure of dependence for i.d. processes. As shown in [26], it carries enough information to detect ergodic properties of $Y(t)$. The next result establishes the relationship between the asymptotic behaviour of $\tau(t)$ and $C_l(0, t)$.

Theorem 3. *Let $Y(t)$ be a stationary i.d. process of the form (3.7). If the Lévy measure ν_0 of $Y(0)$ has no atoms in $2\pi\mathbb{Z}$, then the following two conditions are equivalent*

- (i) $\lim_{t \rightarrow \infty} C_l(0, t) = 0$ for every $l > 0$,
- (ii) $\lim_{t \rightarrow \infty} \tau(t) = 0$.

PROOF. Theorem 2 yields the equivalence of (i) and mixing. From [26], Theorem 1, we get that condition (ii) is equivalent to mixing in case when the Lévy measure ν_0 of $Y(0)$ has no atoms in $2\pi\mathbb{Z}$. Thus, conditions (i) and (ii) must be equivalent. ■

In what follows, we show, how to modify the obtained results in order to characterize ergodicity and weak mixing. Let us remind that for the class of i.d. stationary processes these two properties are equivalent, [5].

As already discussed in [26], the Maruyama's theorem and its revised version (Theorem 1) carry over to the case of weak mixing if one replaces the convergence on the whole set \mathbb{R} to the convergence on a subset of density one. Let us remind that a set $D \subset \mathbb{R}_+$ is of density one if $\lim_{C \rightarrow \infty} \lambda(D \cap [0, C])/C = 1$. Here λ denotes the Lebesgue measure. Thus, the version of Theorem 1 for weak mixing has the form

Theorem 4. *An i.d. stationary process $Y(t)$ is weakly mixing (ergodic) if and only if for some set D of density one the following two conditions hold*

- (C1) *the covariance function $\text{Cov}(t)$ of its Gaussian part converges to 0 as $t \rightarrow \infty$, $t \in D$,*
- (C2) $\lim_{t \rightarrow \infty, t \in D} \nu_{0t}(|xy| > \delta) = 0$ *for every $\delta > 0$, where ν_{0t} is the Lévy measure of $(Y(0), Y(t))$.*

Since the intersection of finite number of sets of density one is still the set of density one, we can repeat the arguments of Lemma 2 and Theorem 2 restricted to a set of density one. Hence, we obtain

Theorem 5. *Let $Y(t)$ be a stationary i.d. process of the form (3.7) with no Gaussian part. Then $Y(t)$ is weakly mixing (ergodic) iff for some set D of density one, the corresponding function C_l satisfies*

$$\lim_{t \rightarrow \infty, t \in D} C_l(0, t) = 0$$

for every $l > 0$.

Since, for a nonnegative and bounded function $f : \mathbb{R}_+ \rightarrow \mathbb{R}$ and for a set D of density one, the condition

$$\lim_{t \rightarrow \infty, t \in D} f(t) = 0$$

is equivalent to the following one

$$\lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T f(u) du = 0,$$

hence, we obtain the following corollary

Corollary 4. *Let $Y(t)$ be a stationary i.d. process of the form (3.7) with no Gaussian part. Then $Y(t)$ is weakly mixing (ergodic) iff for some set D of density one, the corresponding function C_l satisfies*

$$\lim_{t \rightarrow \infty} \frac{1}{T} \int_0^T C_l(0, t) dt = 0$$

for every $l > 0$.

Thus, the above results give the description of ergodicity weak mixing and mixing in the language of the function C_l . They indicate that correlation cascade is an appropriate mathematical tool for detecting ergodic properties of i.d. processes. Moreover, Theorem 3 yields the relationship between both measures of dependence $C_l(0, t)$ and $\tau(t)$.

Chapter 4

Codifference and the dependence structure

In Section 2.2, we discussed the properties and the presence of long range dependence in four fractional generalizations of the classical Gaussian O-U process. In this chapter, we extend these investigations to the more general α -stable case. We introduce five stationary α -stable models and study their dependence structure in the language of codifference.

Through the analogy to the Gaussian case (see Section 2.2), the α -stable **O-U process** $\{Z(t), t \in \mathbb{R}\}$ can be equivalently defined as:

- (a) the Lamperti transformation of the symmetric α -stable Lévy motion $L_\alpha(t)$, $0 < \alpha \leq 2$ (see [11])

$$Z(t) = e^{-\lambda t} L_\alpha(e^{\alpha \lambda t}) \quad (4.1)$$

- (b) the stationary solution of the α -stable Langevin equation

$$\frac{dZ(t)}{dt} + \lambda Z(t) = l_\alpha(t), \quad (4.2)$$

where $l_\alpha(t)$ is the α -stable noise, i.e. $l_\alpha(t) = dL_\alpha(t)/dt$.

The integral representation of $Z(t)$ is given by

$$Z(t) = \int_{-\infty}^t e^{-\lambda(t-s)} dL_\alpha(s), \quad (4.3)$$

which immediately implies that $Z(t)$ is stationary and Markovian. Moreover, it has α -stable marginal distributions and for $\alpha = 2$ we recover the classical Gaussian O-U process. As shown in [29], the codifference of $Z(t)$ decays exponentially. This result is analogous to the one for correlations in the Gaussian case, and indicates that the α -stable O-U process has short memory in the sense of (3.3).

In what follows, we define four fractional generalizations of $Z(t)$, explore their dependence structure and answer the question of long memory in these models.

4.1 Type I fractional α -stable Ornstein-Uhlenbeck process

In this section we define the first generalization $\{Z_1(t), t \in \mathbb{R}\}$ of the standard O-U process. $Z_1(t)$ is the α -stable extension of the Gaussian process $Y_1(t)$, see (2.8).

Definition 5. Let $L_{\alpha,H}(t)$, $0 < \alpha \leq 2$, $0 < H < 1$, be the fractional α -stable motion (3.5). Then, the process defined as the following Lamperti transformation

$$Z_1(t) = e^{-tH} L_{\alpha,H}(e^t). \quad (4.4)$$

is called **Type I fractional α -stable Ornstein-Uhlenbeck process**.

In the next three theorems we give precise formulas for the asymptotic behaviour of the generalized codifference $I(\theta_1; \theta_2; t)$ introduced in (3.4). Next, we show that similarly to the Gaussian case the process $Z_1(t)$ has short memory. In our considerations, we exclude the two case $\theta_1 \theta_2 = 0$, since then, trivially, $I(\theta_1; \theta_2; t) = 0$.

In the proofs we frequently use the following property ([29], page 122)

$$E \left[\exp \left\{ i\theta \int_B f(x) L_\alpha(dx) \right\} \right] = \exp \left\{ -|\theta|^\alpha \int_B |f(x)|^\alpha dx \right\} \quad (4.5)$$

with $B \subset \mathbb{R}$ and $f \in L^\alpha((B), dx)$. We also take advantage of the two key inequalities [15]:

For $r, s \in \mathbb{R}$

$$||r + s|^\alpha - |r|^\alpha - |s|^\alpha| \leq \begin{cases} 2|r|^\alpha & \text{if } 0 < \alpha \leq 1 \\ (\alpha + 1)|r|^\alpha + \alpha|r||s|^{\alpha-1} & \text{if } 1 < \alpha \leq 2. \end{cases} \quad (4.6)$$

Theorem 6. Let $0 < \alpha < 1$ and $0 < H < 1$. Then the generalized codifference of $Z_1(t)$ satisfies

$$I(\theta_1; \theta_2; t) \sim A_\alpha(\theta_1; \theta_2) e^{-t\alpha H(1-H)}$$

as $t \rightarrow \infty$, where

$$\begin{aligned} A_\alpha(\theta_1; \theta_2) &= \int_0^\infty \left\{ | -\theta_1 a s^{H-1/\alpha} + \theta_2 a(H-1/\alpha) s^{H-1/\alpha-1} |^\alpha \right. \\ &\quad \left. - | \theta_1 a s^{H-1/\alpha} |^\alpha - | \theta_2 a(H-1/\alpha) s^{H-1/\alpha-1} |^\alpha \right\} ds \\ &\quad + \int_0^\infty \left\{ | -\theta_1 b s^{H-1/\alpha} + \theta_2 b(1/\alpha - H) s^{H-1/\alpha-1} |^\alpha \right. \\ &\quad \left. - | \theta_1 b s^{H-1/\alpha} |^\alpha - | \theta_2 b(1/\alpha - H) s^{H-1/\alpha-1} |^\alpha \right\} ds. \end{aligned} \quad (4.7)$$

PROOF: We have $Z_1(t) = e^{-tH} L_{\alpha,H}(e^t) = \int_{-\infty}^\infty f(s, t) L_\alpha(ds)$ with

$$f(s, t) = e^{-tH} a \left[(e^t - s)_+^{H-1/\alpha} - (-s)_+^{H-1/\alpha} \right] + e^{-tH} b \left[(e^t - s)_-^{H-1/\alpha} - (-s)_-^{H-1/\alpha} \right].$$

Taking advantage of (3.4) and (4.5) we obtain

$$\begin{aligned}
I(\theta_1; \theta_2; t) &= \int_{-\infty}^{\infty} \{|\theta_1 f(s, t) + \theta_2 f(s, 0)|^\alpha - |\theta_1 f(s, t)|^\alpha - |\theta_2 f(s, 0)|^\alpha\} ds \\
&= \int_{-\infty}^0 \dots ds + \int_0^1 \dots ds + \int_1^{e^t} \dots ds + \int_{e^t}^{\infty} \dots ds \\
&=: I_1(t) + I_2(t) + I_3(t) + I_4(t).
\end{aligned} \tag{4.8}$$

In what follows, we estimate every $I_j(t)$, $j = 1, \dots, 4$, separately.

Let us begin with $I_1(t)$. After some standard calculations and by the change of variables $s \rightarrow -e^{tH}s$, we get

$$I_1(t) = e^{t\alpha H^2} \int_0^{\infty} \{|p(s, t) + q(s, t)|^\alpha - |p(s, t)|^\alpha - |q(s, t)|^\alpha\} ds,$$

where

$$p(s, t) = e^{-tH} \theta_1 a [(e^{t(1-H)} + s)^{H-1/\alpha} - s^{H-1/\alpha}] \quad \text{and} \tag{4.9}$$

$$q(s, t) = \theta_2 a [(e^{-tH} + s)^{H-1/\alpha} - s^{H-1/\alpha}]. \tag{4.10}$$

For fixed $s \in (0, \infty)$ we see that

$$e^{tH} p(s, t) \longrightarrow -\theta_1 a s^{H-1/\alpha} =: p_\infty(s)$$

as $t \rightarrow \infty$. Using the mean-value theorem

$$f(r+s) - f(r) = s \int_0^1 f'(r+us) du, \tag{4.11}$$

where f is accordingly smooth, and the dominated convergence theorem, we obtain

$$e^{tH} q(s, t) = \theta_2 a (H-1/\alpha) \int_0^1 (s+ue^{-tH})^{H-1/\alpha-1} du \longrightarrow \theta_2 a (H-1/\alpha) s^{H-1/\alpha-1} =: q_\infty(s)$$

as $t \rightarrow \infty$. Consequently, for fixed $s \in (0, \infty)$

$$\begin{aligned}
e^{t\alpha H} \{|p(s, t) + q(s, t)|^\alpha - |p(s, t)|^\alpha - |q(s, t)|^\alpha\} &\longrightarrow \\
\{|p_\infty(s) + q_\infty(s)|^\alpha - |p_\infty(s)|^\alpha - |q_\infty(s)|^\alpha\}
\end{aligned} \tag{4.12}$$

as $t \rightarrow \infty$. To apply the dominated convergence theorem, we use inequality (4.6) together with the mean-value theorem and get

$$\begin{aligned}
&\sup_{t>1} e^{t\alpha H} ||p(s, t) + q(s, t)|^\alpha - |p(s, t)|^\alpha - |q(s, t)|^\alpha| \\
&\leq \sup_{t>1} \mathbf{1}_{(0,1)}(s) e^{t\alpha H} ||p(s, t) + q(s, t)|^\alpha - |p(s, t)|^\alpha - |q(s, t)|^\alpha| \\
&\quad + \sup_{t>1} \mathbf{1}_{[1,\infty)}(s) e^{t\alpha H} ||p(s, t) + q(s, t)|^\alpha - |p(s, t)|^\alpha - |q(s, t)|^\alpha| \\
&\leq \sup_{t>1} \mathbf{1}_{(0,1)}(s) e^{t\alpha H} 2|p(s, t)|^\alpha + \sup_{t>1} \mathbf{1}_{[1,\infty)}(s) e^{t\alpha H} 2|q(s, t)|^\alpha \\
&\leq \mathbf{1}_{(0,1)}(s) c_1 s^{H\alpha-1} + \mathbf{1}_{[1,\infty)}(s) c_2 s^{H\alpha-1-\alpha},
\end{aligned}$$

which is integrable on $(0, \infty)$. Here c_1 and c_2 are the appropriate constants independent of s and t . Thus, from the dominated convergence theorem we get

$$I_1(t) \sim e^{t\alpha H^2} e^{-t\alpha H} \int_0^\infty \{|p_\infty(s) + q_\infty(s)|^\alpha - |p_\infty(s)|^\alpha - |q_\infty(s)|^\alpha\} ds \quad (4.13)$$

as $t \rightarrow \infty$.

We pass on to $I_2(t) = \int_0^1 \{|v(s, t) + u(s)|^\alpha - |v(s, t)|^\alpha - |u(s)|^\alpha\} ds$, where

$$v(s, t) = e^{-tH} \theta_1 [a(e^t - s)^{H-1/\alpha} - bs^{H-1/\alpha}] \text{ and}$$

$$u(s) = \theta_2 [a(1-s)^{H-1/\alpha} - bs^{H-1/\alpha}].$$

>From (4.6) we obtain $|I_2(t)| \leq 2 \int_0^1 |v(s, t)|^\alpha$. Additionally, for fixed $s \in (0, 1)$ we have

$$e^{tH} v(s, t) \longrightarrow -\theta_1 bs^{H-1/\alpha}$$

as $t \rightarrow \infty$, and

$$\sup_{t>1} e^{t\alpha H} |v(s, t)|^\alpha \leq d_1(1-s)^{H\alpha-1} + d_2 s^{H\alpha-1},$$

which is integrable on $(0, 1)$. Here d_1 and d_2 are the appropriate constants independent of s and t . Thus, we obtain $I_2(t) = O(e^{-t\alpha H})$, which implies

$$e^{t\alpha H(1-H)} I_2(t) \longrightarrow 0 \text{ as } t \rightarrow \infty$$

and the contribution of $I_2(t)$ is negligible.

We continue our estimations for $I_3(t)$. After the change of variables $s \rightarrow e^{tH} s$, we have

$$I_3(t) = e^{t\alpha H^2} \int_0^\infty \{|w(s, t) + z(s, t)|^\alpha - |w(s, t)|^\alpha - |z(s, t)|^\alpha\} ds,$$

where

$$w(s, t) = e^{-tH} \theta_1 [a(e^{t(1-H)} - s)^{H-1/\alpha} - bs^{H-1/\alpha}] \cdot \mathbf{1}_{(e^{-tH}, e^{t(1-H)})}(s) \quad (4.14)$$

and

$$z(s, t) = \theta_2 b[(s - e^{-tH})^{H-1/\alpha} - s^{H-1/\alpha}] \cdot \mathbf{1}_{(e^{-tH}, e^{t(1-H)})}(s). \quad (4.15)$$

In a similar manner as for $I_1(t)$, we get that for fixed $s \in (0, \infty)$

$$e^{tH} w(s, t) \longrightarrow -\theta_1 bs^{H-1/\alpha} =: w_\infty(s)$$

and also

$$e^{tH} z(s, t) \longrightarrow \theta_2 b(1/\alpha - H) s^{H-1/\alpha-1} =: z_\infty(s)$$

as $t \rightarrow \infty$. Consequently,

$$e^{t\alpha H} \{|w(s, t) + z(s, t)|^\alpha - |w(s, t)|^\alpha - |z(s, t)|^\alpha\} \longrightarrow \{|w_\infty(s) + z_\infty(s)|^\alpha - |w_\infty(s)|^\alpha - |z_\infty(s)|^\alpha\}$$

as $t \rightarrow \infty$. Using (4.6) we obtain the following estimate

$$\begin{aligned}
& \sup_{t > \frac{2}{H}} e^{t\alpha H} ||w(s, t) + z(s, t)|^\alpha - |w(s, t)|^\alpha - |z(s, t)|^\alpha| \\
& \leq \sup_{t > \frac{2}{H}} \mathbf{1}_{(0,1)}(s) e^{t\alpha H} 2|w(s, t)|^\alpha + \sup_{t > \frac{2}{H}} \mathbf{1}_{[1,\infty)}(s) e^{t\alpha H} 2|z(s, t)|^\alpha \\
& \leq \mathbf{1}_{(0,1)}(s) [e_1(1-s)^{H\alpha-1} + e_2 s^{H\alpha-1}] + \mathbf{1}_{[1,\infty)}(s) e_3 (s-1/2)^{H\alpha-1-\alpha},
\end{aligned}$$

which is integrable on $(0, \infty)$. Here e_1 , e_2 and e_3 are the appropriate constants independent of s and t . Therefore, the dominated convergence theorem yields

$$I_3(t) \sim e^{t\alpha H^2} e^{-t\alpha H} \int_0^\infty \{|w_\infty(s) + z_\infty(s)|^\alpha - |w_\infty(s)|^\alpha - |z_\infty(s)|^\alpha\} ds \quad (4.16)$$

as $t \rightarrow \infty$.

For $I_4(t)$, after the change of variables $s \rightarrow e^t s$, we get

$$I_4(t) = e^{t\alpha H} \int_1^\infty \{|g(s, t) + h(s, t)|^\alpha - |g(s, t)|^\alpha - |h(s, t)|^\alpha\} ds,$$

with

$$g(s, t) = e^{-tH} \theta_1 b[(s-1)^{H-1/\alpha} - s^{H-1/\alpha}]$$

and

$$h(s, t) = \theta_2 b[(s - e^{-t})^{H-1/\alpha} - s^{H-1/\alpha}].$$

Form the mean-value theorem we get

$$h(s, t) = \theta_2 b(1/\alpha - H) e^{-t} \int_0^1 (s - e^{-tu})^{H-1/\alpha-1} du,$$

therefore, for fixed $s \in (1, \infty)$ we have $e^t h(s, t) \rightarrow \theta_2 b(1/\alpha - H) s^{H-1/\alpha-1}$ as $t \rightarrow \infty$. Additionally

$$\sup_{t > 2} e^{t\alpha} |h(s, t)|^\alpha \leq |\theta_2 b(1/\alpha - H)|^\alpha (s - 1/2)^{H\alpha-1-\alpha},$$

which is integrable on $(1, \infty)$. Since $|I_4(t)| \leq 2e^{t\alpha H} \int_1^\infty |h(s, t)|^\alpha ds$, we obtain $I_4(t) = O(e^{-t\alpha(1-H)})$, thus the contribution of $I_4(t)$, similarly to $I_2(t)$, is negligible. Finally, putting together formulas (4.13) and (4.16), we get the desired result. ■

We pass on to the case $\alpha = 1$. We recall the fact that the case $\theta_1 \theta_2 = 0$ is excluded, since then $I(\theta_1; \theta_2; t) = 0$.

Theorem 7. *Let $\alpha = 1$ and $0 < H < 1$. Then the generalized codifference of $Z_1(t)$ satisfies*

(i) *If $b = 0$ and $\theta_1 \theta_2 > 0$ then $I(\theta_1; \theta_2; t) = 0$,*

(ii) If $a = 0$ and $\theta_1\theta_2 < 0$ then

$$I(\theta_1; \theta_2; t) \sim -2|b| \left(\frac{|\theta_1|}{H} e^{-tH} \cdot \mathbf{1}_{(0,1/2]}(H) + |\theta_2| e^{-t(1-H)} \cdot \mathbf{1}_{[1/2,1)}(H) \right)$$

as $t \rightarrow \infty$.

Otherwise

$$I(\theta_1; \theta_2; t) \sim A_1(\theta_1; \theta_2) e^{-tH(1-H)}$$

as $t \rightarrow \infty$, where A_1 is given in (4.7).

PROOF: First, we determine, in which case the constant $A_1(\theta_1; \theta_2) = 0$. From (4.7) we get

$$A_1(\theta_1; \theta_2) = \int_0^\infty \dots ds + \int_0^\infty \dots ds =: A_{11}(\theta_1; \theta_2) + A_{12}(\theta_1; \theta_2).$$

>From the triangle inequality we see that $A_1(\theta_1; \theta_2) = 0 \Leftrightarrow \{A_{11}(\theta_1; \theta_2) = 0 \text{ and } A_{12}(\theta_1; \theta_2) = 0\}$. Additionally, we have $A_{11}(\theta_1; \theta_2) = 0 \Leftrightarrow \{a = 0 \text{ or } \theta_1\theta_2 > 0\}$ as well as $A_{12}(\theta_1; \theta_2) = 0 \Leftrightarrow \{b = 0 \text{ or } \theta_1\theta_2 < 0\}$. Since the cases $\theta_1\theta_2 = 0$ or $a = b = 0$ are excluded, we obtain

$$A_1(\theta_1; \theta_2) = 0 \Leftrightarrow \{a = 0 \text{ and } \theta_1\theta_2 < 0\} \text{ or } \{b = 0 \text{ and } \theta_1\theta_2 > 0\}.$$

The case $\{b = 0 \text{ and } \theta_1\theta_2 > 0\}$ is trivial, since then it is easy to verify that for every term in formula (4.8) we have $I_j(t) = 0$, $i = 1, \dots, 4$. Thus, we obtain part (i) of the theorem.

We pass on to the second possibility $\{a = 0 \text{ and } \theta_1\theta_2 < 0\}$. In this case only $I_1(t)$ and $I_3(t)$ from (4.8) disappear, therefore we need to find the asymptotic behaviour of $I_2(t)$ and $I_4(t)$. Let us begin with $I_2(t)$. From the proof of Th.6 we get $I_2(t) = \int_0^1 \{|v(s, t) + u(s)| - |v(s, t)| - |u(s)|\} ds$, where $v(s, t) = -e^{-tH} \theta_1 b s^{H-1}$ and $u(s) = -\theta_2 b s^{H-1}$. First, we consider the case $\theta_1 > 0$ and $\theta_2 < 0$. Fix $s \in (0, 1)$. Then for large enough t we get

$$|v(s, t) + u(s)| - |v(s, t)| - |u(s)| = -2e^{-tH} |b| \theta_1 s^{H-1},$$

which implies $e^{tH} [|v(s, t) + u(s)| - |v(s, t)| - |u(s)|] \longrightarrow -2|b| \theta_1 s^{H-1}$ as $t \rightarrow \infty$. Since

$$\sup_{t>1} e^{tH} ||v(s, t) + u(s)| - |v(s, t)| - |u(s)|| \leq 2|b| \theta_1 s^{H-1},$$

we get from the dominated convergence theorem $I_2(t) \sim -2|b| \theta_1 e^{-tH} \int_0^1 s^{H-1} ds = -2|b| \frac{\theta_1}{H} e^{-tH}$ as $t \rightarrow \infty$. Symmetrically, for $\theta_1 < 0$ and $\theta_2 > 0$ one can show that $I_2(t) \sim 2|b| \frac{\theta_1}{H} e^{-tH}$. Finally

$$I_2(t) \sim -2|b| \frac{|\theta_1|}{H} e^{-tH} \quad (4.17)$$

as $t \rightarrow \infty$.

We continue with $I_4(t)$. From the proof of Th.6 we have

$$I_4(t) = e^{tH} \int_1^\infty \{|g(s, t) + h(s, t)| - |g(s, t)| - |h(s, t)|\} ds,$$

with

$$g(s, t) = e^{-tH} \theta_1 b [(s-1)^{H-1} - s^{H-1}]$$

and

$$h(s, t) = \theta_2 b [(s - e^{-t})^{H-1} - s^{H-1}].$$

For fixed $s \in (1, \infty)$ we have $e^{tH} g(s, t) \rightarrow \theta_1 b [(s-1)^{H-1} - s^{H-1}]$, and also $e^t h(s, t) \rightarrow \theta_2 b (1-H) s^{H-2}$ as $t \rightarrow \infty$, which implies that for large enough t we get $|g(s, t)| > |h(s, t)|$. Let us then consider the case $\theta_1 > 0$ and $\theta_2 < 0$. For fixed $s \in (1, \infty)$ and large t we obtain

$$|g(s, t) + h(s, t)| - |g(s, t)| - |h(s, t)| = 2\theta_2 |b| [(s - e^{-t})^{H-1} - s^{H-1}],$$

and consequently

$$e^t \{|g(s, t) + h(s, t)| - |g(s, t)| - |h(s, t)|\} \rightarrow 2\theta_2 |b| (1-H) s^{H-2}$$

as $t \rightarrow \infty$. We also have from (4.6)

$$\sup_{t>2} e^t ||g(s, t) + h(s, t)| - |g(s, t)| - |h(s, t)|| \leq k_1 (s-1/2)^{H-2},$$

which is integrable on $(1, \infty)$. Here k_1 is the appropriate constant independent of s and t . Thus, from the dominated convergence theorem we get $I_4(t) \sim e^{-t(1-H)} \cdot 2\theta_2 |b| (1-H) \int_1^\infty s^{H-2} ds = e^{-t(1-H)} \cdot 2\theta_2 |b|$ as $t \rightarrow \infty$. For $\theta_1 < 0$ and $\theta_2 > 0$ one shows in a similar manner that $I_4(t) \sim -e^{-t(1-H)} \cdot 2\theta_2 |b|$. Finally

$$I_4(t) \sim -2|\theta_2 b| e^{-t(1-H)} \quad (4.18)$$

as $t \rightarrow \infty$.

Now, from (4.17) and (4.18) we get that for $H < 1/2$ we obtain $I(\theta_1; \theta_2; t) \sim I_2(t)$, for $H > 1/2$ we obtain $I(\theta_1; \theta_2; t) \sim I_4(t)$ and for $H = 1/2$ we get $I(\theta_1; \theta_2; t) \sim I_2(t) + I_4(t)$ as $t \rightarrow \infty$. Thus, we have proved part (ii) of the theorem.

In any other case, i.e. when $A_1(\theta_1; \theta_2)$ is a non-zero constant, the proof of Theorem 6 applies and we get $I(\theta_1; \theta_2; t) \sim A_1(\theta_1; \theta_2) e^{-tH(1-H)}$ as $t \rightarrow \infty$. ■

The next theorem determines the asymptotic dependence structure of $Z_1(t)$ when the index of stability is such that $1 < \alpha < 2$.

Theorem 8. *Let $1 < \alpha < 2$, $0 < H < 1$ and $H \neq 1/\alpha$. Then the generalized codifference of $Z_1(t)$ satisfies*

(i) If $1 - \frac{1}{\alpha} < H < \frac{1}{\alpha}$ then

$$I(\theta_1; \theta_2; t) \sim A_\alpha(\theta_1; \theta_2)e^{-t\alpha H(1-H)}$$

(ii) If $H < 1 - \frac{1}{\alpha}$ then

$$I(\theta_1; \theta_2; t) \sim B_\alpha(\theta_1; \theta_2)e^{-tH}$$

(iii) If $H > \frac{1}{\alpha}$ then

$$I(\theta_1; \theta_2; t) \sim D_\alpha(\theta_1; \theta_2)e^{-t(1-H)}$$

as $t \rightarrow \infty$. The constant A_α is given in (4.7), whereas

$$\begin{aligned} B_\alpha(\theta_1; \theta_2) &= \int_0^\infty \alpha \theta_1 \operatorname{sgn}\{\theta_2\} |\theta_2|^{\alpha-1} |a|^\alpha \left| (1+s)^{H-1/\alpha} - s^{H-1/\alpha} \right|^{\alpha-1} s^{H-1/\alpha} ds \\ &\quad - \int_0^1 \alpha \theta_1 b s^{H-1/\alpha} \left| \theta_2 [a(1-s)^{H-1/\alpha} - b s^{H-1/\alpha}] \right|^{\alpha-1} \times \\ &\quad \operatorname{sgn}\left\{ \theta_2 [a(1-s)^{H-1/\alpha} - b s^{H-1/\alpha}] \right\} ds \\ &\quad - \int_1^\infty \alpha \theta_1 \operatorname{sgn}\{\theta_2 b\} b s^{H-1/\alpha} \left| \theta_2 b [(s-1)^{H-1/\alpha} - s^{H-1/\alpha}] \right|^{\alpha-1} ds, \end{aligned}$$

and

$$\begin{aligned} D_\alpha(\theta_1; \theta_2) &= |a|^\alpha \int_0^\infty \alpha (H - 1/\alpha) \theta_2 |\theta_1|^{\alpha-1} \operatorname{sgn}\{\theta_1\} s^{H-1/\alpha-1} |(s+1)^{H-1/\alpha} - s^{H-1/\alpha}|^{\alpha-1} ds \\ &\quad + \int_0^1 \alpha |\theta_1|^{\alpha-1} \theta_2 b (1/\alpha - H) s^{H-1/\alpha-1} |a(1-s)^{H-1/\alpha} - b s^{H-1/\alpha}|^{\alpha-1} \times \\ &\quad \operatorname{sgn}\{\theta_1 [a(1-s)^{H-1/\alpha} - b s^{H-1/\alpha}]\} ds \\ &\quad + \int_1^\infty \alpha \theta_2 b (H - 1/\alpha) \operatorname{sgn}\{\theta_1 b\} |\theta_1 b|^{\alpha-1} |(s-1)^{H-1/\alpha} - s^{H-1/\alpha}|^{\alpha-1} s^{H-1/\alpha-1} ds. \end{aligned}$$

PROOF:

(i) Recall the decomposition in (4.8). We have $I_1(t) = e^{t\alpha H^2} \int_0^\infty \{|p(s, t) + q(s, t)|^\alpha - |p(s, t)|^\alpha - |q(s, t)|^\alpha\} ds$, where $p(s, t)$ and $q(s, t)$ are given in (4.9) and (4.10), respectively. For fixed $s \in (0, \infty)$ we get that (4.12) holds, since $H - 1/\alpha < 0$. To apply the dominated convergence theorem, we use the second part of inequality (4.6) in combination with the mean-value theorem and get

$$\begin{aligned} &\sup_{t>1} e^{t\alpha H} ||p(s, t) + q(s, t)|^\alpha - |p(s, t)|^\alpha - |q(s, t)|^\alpha| \\ &\leq \sup_{t>1} \mathbf{1}_{(0,1)}(s) e^{t\alpha H} ||p(s, t) + q(s, t)|^\alpha - |p(s, t)|^\alpha - |q(s, t)|^\alpha| \\ &\quad + \sup_{t>1} \mathbf{1}_{[1,\infty)}(s) e^{t\alpha H} ||p(s, t) + q(s, t)|^\alpha - |p(s, t)|^\alpha - |q(s, t)|^\alpha| \\ &\leq \sup_{t>1} \mathbf{1}_{(0,1)}(s) e^{t\alpha H} (\alpha + 1) |p(s, t)|^\alpha + \sup_{t>1} \mathbf{1}_{(0,1)}(s) e^{t\alpha H} \alpha |p(s, t)| |q(s, t)|^{\alpha-1} \\ &\quad + \sup_{t>1} \mathbf{1}_{[1,\infty)}(s) e^{t\alpha H} (\alpha + 1) |q(s, t)|^\alpha + \sup_{t>1} \mathbf{1}_{[1,\infty)}(s) e^{t\alpha H} \alpha |q(s, t)| |p(s, t)|^{\alpha-1} \\ &\leq \mathbf{1}_{(0,1)}(s) c_1 s^{H\alpha-1} + \mathbf{1}_{(0,1)}(s) c_2 s^{H\alpha-\alpha} + \mathbf{1}_{[1,\infty)}(s) c_3 s^{H\alpha-1-\alpha} + \mathbf{1}_{[1,\infty)}(s) c_4 s^{H\alpha-2}, \end{aligned}$$

which is integrable on $(0, \infty)$, since $H\alpha < 1$ and $H\alpha - \alpha > -1$. Here c_i , $i = 1, \dots, 4$, are the appropriate constants independent of s and t . Therefore, we get

$$I_1(t) \sim e^{t\alpha H^2} e^{-t\alpha H} \int_0^\infty \{|p_\infty(s) + q_\infty(s)|^\alpha - |p_\infty(s)|^\alpha - |q_\infty(s)|^\alpha\} ds \quad (4.19)$$

as $t \rightarrow \infty$.

Next, we estimate $I_2(t) = \int_0^1 \{|v(s, t) + u(s)|^\alpha - |v(s, t)|^\alpha - |u(s)|^\alpha\} ds$, with $v(s, t) = e^{-tH} \theta_1 [a(e^t - s)^{H-1/\alpha} - bs^{H-1/\alpha}]$ and $u(s) = \theta_2 [a(1-s)^{H-1/\alpha} - bs^{H-1/\alpha}]$. From (4.6) we obtain

$$|I_2(t)| \leq (\alpha + 1) \int_0^1 |v(s, t)|^\alpha ds + \alpha \int_0^1 |v(s, t)| |u(s)|^{\alpha-1} ds.$$

Additionally, for fixed $s \in (0, 1)$ we have $e^{t\alpha H} |v(s, t)|^\alpha \rightarrow |b\theta_1|^\alpha s^{H\alpha-1}$ and $e^{tH} |v(s, t)| |u(s)|^{\alpha-1} \rightarrow |b\theta_1| s^{H-1/\alpha} |u(s)|^{\alpha-1}$ as $t \rightarrow \infty$. We also get

$$\sup_{t>1} e^{t\alpha H} |v(s, t)|^\alpha \leq d_1 (1-s)^{H\alpha-1} + d_2 s^{H\alpha-1}$$

and

$$\sup_{t>1} e^{tH} |v(s, t)| |u(s)|^{\alpha-1} \leq e_1 (1-s)^{H\alpha-1} + e_2 s^{H\alpha-1},$$

where d_1 , d_2 , e_1 and e_2 are the appropriate constants independent of s and t . Thus, from the dominated convergence theorem we get $I_2(t) = O(e^{-tH} + e^{-t\alpha H}) = O(e^{-tH})$, $t \rightarrow \infty$. Thus, since $\alpha(1-H) < 1$, the integral $I_2(t)$ decays faster than $I_1(t)$.

In case of $I_3(t)$ we can not use the dominated convergence theorem directly, we need more delicate estimations. We have $I_3(t) = e^{t\alpha H^2} \int_0^\infty \{|w(s, t) + z(s, t)|^\alpha - |w(s, t)|^\alpha - |z(s, t)|^\alpha\} ds$, where $w(s, t)$ and $z(s, t)$ are given in (4.14) and (4.15), respectively. Set

$$G(s, t) := |w(s, t) + z(s, t)|^\alpha - |w(s, t)|^\alpha - |z(s, t)|^\alpha,$$

$$G_\infty(s) := |w_\infty(s) + z_\infty(s)|^\alpha - |w_\infty(s)|^\alpha - |z_\infty(s)|^\alpha$$

with $w_\infty(s) = -\theta_1 bs^{H-1/\alpha}$ and $z_\infty(s) = \theta_2 b(1/\alpha - H)s^{H-1/\alpha-1}$. We will show that

$$\left| \int_0^\infty [e^{t\alpha H} G(s, t) - G_\infty(s)] ds \right| \rightarrow 0$$

as $t \rightarrow \infty$. Fix $\epsilon > 0$ and put

$$\begin{aligned} & \int_0^\infty [e^{t\alpha H} G(s, t) - G_\infty(s)] ds = \\ &= \int_0^{e^{-tH}} \dots ds + \int_{e^{-tH}}^{e^{-tH} + \epsilon} \dots ds + \int_{e^{-tH} + \epsilon}^{e^{t(1-H)} - \epsilon} \dots ds + \int_{e^{t(1-H)} - \epsilon}^{e^{t(1-H)}} \dots ds + \int_{e^{t(1-H)}}^\infty \dots ds \\ &=: J_1(t) + J_2(t) + J_3(t) + J_4(t) + J_5(t). \end{aligned}$$

Since $G_\infty(s)$ is integrable on $(0, \infty)$, we obtain $|J_1(t)| \rightarrow 0$ and $|J_5(t)| \rightarrow 0$ as $t \rightarrow \infty$. For the second term we have from (4.6) that

$$\begin{aligned}
|J_2(t)| &\leq \int_{e^{-tH}}^{e^{-tH}+\epsilon} |e^{t\alpha H} G(s, t) - G_\infty(s)| ds \leq (\alpha + 1) \int_{e^{-tH}}^{e^{-tH}+\epsilon} e^{t\alpha H} |w(s, t)|^\alpha ds \\
&+ \alpha \int_{e^{-tH}}^{e^{-tH}+\epsilon} e^{t\alpha H} |w(s, t)| |z(s, t)|^{\alpha-1} ds + (\alpha + 1) \int_{e^{-tH}}^{e^{-tH}+\epsilon} |w_\infty(s)|^\alpha ds \\
&+ \alpha \int_{e^{-tH}}^{e^{-tH}+\epsilon} |w_\infty(s)| |z_\infty(s)|^{\alpha-1} ds =: J_{21}(t) + J_{22}(t) + J_{23}(t) + J_{24}(t).
\end{aligned}$$

Additionally, since $H - 1/\alpha < 0$, we have

$$\begin{aligned}
J_{21}(t) &\leq d_1 \int_{e^{-tH}}^{e^{-tH}+\epsilon} [(1-s)^{H\alpha-1} + s^{H\alpha-1}] ds = -\frac{d_1}{H\alpha} [(1-e^{-tH}-\epsilon)^{H\alpha} - (1-e^{-tH})^{H\alpha}] \\
&+ \frac{d_1}{H\alpha} [(e^{-tH}+\epsilon)^{H\alpha} - (e^{-tH})^{H\alpha}].
\end{aligned}$$

Next, since $H - 1/\alpha < 0$, we get for $s \in (e^{-tH}, e^{-tH}+\epsilon)$

$$|(s - e^{-tH})^{H-1/\alpha} - s^{H-1/\alpha}| \leq (1/\alpha - H)e^{-tH}(s - e^{-tH})^{H-1/\alpha-1},$$

and consequently

$$\begin{aligned}
J_{22}(t) &\leq d_2 \int_{e^{-tH}}^{e^{-tH}+\epsilon} [(1-s)^{H-1/\alpha} + s^{H-1/\alpha}](s - e^{-tH})^{(H-1/\alpha-1)(\alpha-1)} ds \\
&\leq d_3(1 - e^{-tH} - \epsilon)^{H-1/\alpha} \epsilon^{H\alpha-\alpha+1/\alpha-H+1} + d_4 \epsilon^{H\alpha-\alpha+1}.
\end{aligned}$$

Moreover,

$$J_{23}(t) = d_5[(e^{-tH} + \epsilon)^{H\alpha} - (e^{-tH})^{H\alpha}],$$

$$J_{24}(t) = d_6[(e^{-tH} + \epsilon)^{H\alpha-\alpha+1} - (e^{-tH})^{H\alpha-\alpha+1}],$$

where d_i , $i = 1, \dots, 6$ are the appropriate constants independent of t , s and ϵ . Using the fact that $1 - \frac{1}{\alpha} < H < \frac{1}{\alpha}$, we obtain $\lim_{\epsilon \searrow 0} \lim_{t \rightarrow \infty} J_{2i}(t) = 0$ for every $i = 1, \dots, 4$, which implies $\lim_{\epsilon \searrow 0} \lim_{t \rightarrow \infty} J_2(t) = 0$.

We pass on to $J_3(t)$. For fixed $s \in (0, \infty)$ we get from the proof of Theorem 6

$$e^{t\alpha H} G(s, t) \mathbf{1}_{(e^{-tH}+\epsilon, e^{t(1-H)}-\epsilon)}(s) \rightarrow G_\infty(s) \mathbf{1}_{(\epsilon, \infty)}(s).$$

as $t \rightarrow \infty$. Additionally,

$$\begin{aligned}
& \sup_{t > \frac{2}{H}} e^{t\alpha H} |G(s, t)| \mathbf{1}_{(e^{-tH} + \epsilon, e^{t(1-H)} - \epsilon)}(s) \\
& \leq \sup_{t > \frac{2}{H}} \mathbf{1}_{(e^{-tH} + \epsilon, 1)}(s) e^{t\alpha H} (\alpha + 1) |w(s, t)|^\alpha + \sup_{t > \frac{2}{H}} \mathbf{1}_{(e^{-tH} + \epsilon, 1)}(s) e^{t\alpha H} \alpha |w(s, t)| |z(s, t)|^{\alpha-1} \\
& \quad + \sup_{t > \frac{2}{H}} \mathbf{1}_{[1, e^{t(1-H)} - \epsilon)}(s) e^{t\alpha H} (\alpha + 1) |z(s, t)|^\alpha + \sup_{t > \frac{2}{H}} \mathbf{1}_{[1, e^{t(1-H)} - \epsilon)}(s) e^{t\alpha H} \alpha |z(s, t)| |w(s, t)|^{\alpha-1} \\
& \leq \mathbf{1}_{(\epsilon, 1)}(s) k_1 [(1-s)^{H\alpha-1} + s^{H\alpha-1}] + \mathbf{1}_{(\epsilon, 1)}(s) k_2 [(1-s)^{H-1/\alpha} + s^{H-1/\alpha}] \epsilon^{(H-1/\alpha-1)(\alpha-1)} \\
& \quad + \mathbf{1}_{[1, \infty)}(s) k_3 (s-1/2)^{H\alpha-1-\alpha} + \mathbf{1}_{[1, \infty)}(s) k_4 (s-1/2)^{H-1/\alpha-1} [\epsilon^{(H-1/\alpha)(\alpha-1)} + s^{(H-1/\alpha)(\alpha-1)}],
\end{aligned}$$

which is integrable on $(0, \infty)$. Thus, from the dominated convergence theorem we get $J_3(t) \rightarrow 0$ as $t \rightarrow \infty$.

For $J_4(t)$ we have

$$\begin{aligned}
|J_4(t)| & \leq \int_{e^{t(1-H)} - \epsilon}^{e^{t(1-H)}} |e^{t\alpha H} G(s, t) - G_\infty(s)| ds \leq (\alpha + 1) \int_{e^{t(1-H)} - \epsilon}^{e^{t(1-H)}} e^{t\alpha H} |z(s, t)|^\alpha ds \\
& \quad + \alpha \int_{e^{t(1-H)} - \epsilon}^{e^{t(1-H)}} e^{t\alpha H} |z(s, t)| |w(s, t)|^{\alpha-1} ds + (\alpha + 1) \int_{e^{t(1-H)} - \epsilon}^{e^{t(1-H)}} |z_\infty(s)|^\alpha ds \\
& \quad + \alpha \int_{e^{t(1-H)} - \epsilon}^{e^{t(1-H)}} |z_\infty(s)| |w_\infty(s)|^{\alpha-1} ds =: J_{41}(t) + J_{42}(t) + J_{43}(t) + J_{44}(t),
\end{aligned}$$

and one can show similarly, as for $J_2(t)$ that $\lim_{\epsilon \searrow 0} \lim_{t \rightarrow \infty} J_4(t) = 0$. Finally, we have proved

$$\lim_{\epsilon \searrow 0} \lim_{t \rightarrow \infty} J_i(t) = 0$$

for $i = 1, \dots, 5$. Thus,

$$\left| \int_0^\infty [e^{t\alpha H} G(s, t) - G_\infty(s)] ds \right| \rightarrow 0,$$

which implies for the term $I_3(t)$ in (4.8) that

$$I_3(t) \sim e^{-t\alpha H(1-H)} \int_0^\infty G_\infty(s) ds$$

as $t \rightarrow \infty$.

For $I_4(t) = e^{t\alpha H} \int_1^\infty \{|g(s, t) + h(s, t)|^\alpha - |g(s, t)|^\alpha - |h(s, t)|^\alpha\} ds$, with $g(s, t) = e^{-tH} \theta_1 b[(s-1)^{H-1/\alpha} - s^{H-1/\alpha}]$ and $h(s, t) = \theta_2 b[(s-e^{-t})^{H-1/\alpha} - s^{H-1/\alpha}]$, we have from (4.6)

$$|I_4(t)| \leq (\alpha + 1) e^{t\alpha H} \int_1^\infty |h(s, t)|^\alpha ds + \alpha e^{t\alpha H} \int_1^\infty |h(s, t)| |g(s, t)|^{\alpha-1} ds.$$

We also obtain from the proof of Theorem 6 that for fixed $s \in (1, \infty)$ we have $e^t h(s, t) \longrightarrow \theta_2 b(1/\alpha - H) s^{H-1/\alpha-1}$ and similarly $e^{tH} g(s, t) \longrightarrow \theta_1 b[(s-1)^{H-1/\alpha} - s^{H-1/\alpha}]$ as $t \rightarrow \infty$. Additionally

$$\sup_{t>2} e^{t\alpha} |h(s, t)|^\alpha \leq |\theta_2 b(1/\alpha - H)|^\alpha (s-1/2)^{H\alpha-1-\alpha}$$

and

$$\sup_{t>2} e^{t+tH(\alpha-1)} |h(s, t)| |g(s, t)|^{\alpha-1} \leq k_1 (s-1/2)^{H-1/\alpha-1} [k_2 (s-1)^{(H-1/\alpha)(\alpha-1)} + k_3 s^{(H-1/\alpha)(\alpha-1)}],$$

which is integrable on $(1, \infty)$. Here k_1 , k_2 and k_3 are the appropriate constants independent of s and t . Thus, $I_4(t) = O(e^{-t(1-H)})$ and, since $\alpha H < 1$, its contribution is negligible.

Finally, we have shown that $I_2(t)$ and $I_4(t)$ decay faster than $I_1(t)$ and $I_3(t)$. Therefore, $I(\theta_1; \theta_2; t) \sim I_1(t) + I_3(t)$ as $t \rightarrow \infty$, which completes the proof of part (i).

(ii) For the first term in (4.8) we have

$$\begin{aligned} I_1(t) &= e^{t\alpha H^2} \int_0^\infty \{|p(s, t) + q(s, t)|^\alpha - |p(s, t)|^\alpha - |q(s, t)|^\alpha\} ds, \\ &= e^{t\alpha H^2} \int_0^1 \dots ds + e^{t\alpha H^2} \int_1^\infty \dots ds =: I_{11}(t) + I_{12}(t). \end{aligned}$$

We recall that $p(s, t) = e^{-tH} \theta_1 a[(e^{t(1-H)} + s)^{H-1/\alpha} - s^{H-1/\alpha}]$ and $q(s, t) = \theta_2 a[(e^{-tH} + s)^{H-1/\alpha} - s^{H-1/\alpha}]$. For $I_{12}(t)$ one shows similarly as in the part (i) of the proof that

$$I_{12}(t) \sim e^{-t\alpha H(1-H)} \int_1^\infty |p_\infty(s) + q_\infty(s)|^\alpha - |p_\infty(s)|^\alpha - |q_\infty(s)|^\alpha ds,$$

as $t \rightarrow \infty$. Here $p_\infty(s) = -\theta_1 a s^{H-1/\alpha}$ and $q_\infty(s) = \theta_2 a (H - 1/\alpha) s^{H-1/\alpha-1}$. For $I_{11}(t)$, after the change of variables $s \rightarrow e^{-tH} s$, we get

$$I_{11}(t) = \int_0^{e^{tH}} \{|\tilde{p}(s, t) + \tilde{q}(s)|^\alpha - |\tilde{p}(s, t)|^\alpha - |\tilde{q}(s)|^\alpha\} ds,$$

where $\tilde{p}(s, t) = e^{-tH} \theta_1 a[(e^t + s)^{H-1/\alpha} - s^{H-1/\alpha}]$ and $\tilde{q}(s) = \theta_2 a[(1 + s)^{H-1/\alpha} - s^{H-1/\alpha}]$. For fixed $s \in (0, \infty)$ we have that $e^{tH} \tilde{p}(s, t) \mathbf{1}_{(0, e^{tH})}(s) \longrightarrow -\theta_1 a s^{H-1/\alpha}$ as $t \rightarrow \infty$, and from the mean-value theorem we obtain

$$\begin{aligned} &e^{tH} \{|\tilde{p}(s, t) + \tilde{q}(s)|^\alpha - |\tilde{q}(s)|^\alpha\} \mathbf{1}_{(0, e^{tH})}(s) \xrightarrow{t \rightarrow \infty} \\ &\alpha \theta_1 \operatorname{sgn}\{\theta_2\} |\theta_2|^{\alpha-1} |a|^\alpha \left| (1 + s)^{H-1/\alpha} - s^{H-1/\alpha} \right|^{\alpha-1} s^{H-1/\alpha} =: H_\infty(s), \end{aligned}$$

since $\frac{d}{dx}|x|^\alpha = \alpha|x|^{\alpha-1}\text{sgn}\{x\}$ for $x \neq 0$. Putting $H(s, t) := |\tilde{p}(s, t) + \tilde{q}(s)|^\alpha - |\tilde{p}(s, t)|^\alpha - |\tilde{q}(s)|^\alpha$ and using inequality (4.6), we get

$$\begin{aligned} \sup_{t>1} e^{tH} |H(s, t)| \mathbf{1}_{(0, e^{tH})}(s) &\leq \sup_{t>1} e^{tH} |H(s, t)| \mathbf{1}_{(0, 1]}(s) + \sup_{t>1} e^{tH} |H(s, t)| \mathbf{1}_{(1, e^{tH})}(s) \\ &\leq \sup_{t>1} (\alpha + 1) e^{tH} |\tilde{p}(s, t)|^\alpha \mathbf{1}_{(0, 1]}(s) + \sup_{t>1} \alpha e^{tH} |\tilde{p}(s, t)| |\tilde{q}(s)|^{\alpha-1} \mathbf{1}_{(0, 1]}(s) \\ &\quad + \sup_{t>1} e^{tH} (\alpha + 1) e^{tH} |\tilde{p}(s, t)|^\alpha \mathbf{1}_{(1, e^{tH})}(s) + \sup_{t>1} e^{tH} \alpha e^{tH} |\tilde{p}(s, t)| |\tilde{q}(s)|^{\alpha-1} \mathbf{1}_{(1, e^{tH})}(s) \\ &\leq l_1 s^{H\alpha-1} \mathbf{1}_{(0, 1]}(s) + l_2 e^{tH-tH\alpha} s^{H\alpha-1} \mathbf{1}_{(1, e^{tH})}(s) + l_3 s^{H\alpha-\alpha} \mathbf{1}_{(1, \infty)}(s) \\ &\leq l_1 s^{H\alpha-1} \mathbf{1}_{(0, 1]}(s) + l_2 s^{H\alpha-\alpha} \mathbf{1}_{(1, \infty)}(s) + l_3 s^{H\alpha-\alpha} \mathbf{1}_{(1, \infty)}(s), \end{aligned}$$

which is integrable on $(0, \infty)$. Here l_i , $i = 1, 2, 3$, are the appropriate constants independent of s and t . Thus, the dominated convergence theorem yields $I_{11}(t) \sim e^{-tH} \int_0^\infty H_\infty(s) ds$ as $t \rightarrow \infty$. Since $\alpha(1-H) > 1$, we see that $I_{12}(t)$ decays faster than $I_{11}(t)$, and we finally obtain

$$I_1(t) \sim e^{-tH} \int_0^\infty H_\infty(s) ds$$

as $t \rightarrow \infty$.

Next, we have $I_2(t) = \int_0^1 \{|v(s, t) + u(s)|^\alpha - |v(s, t)|^\alpha - |u(s)|^\alpha\} ds$, with $v(s, t) = e^{-tH} \theta_1 [a(e^t - s)^{H-1/\alpha} - bs^{H-1/\alpha}]$ and $u(s) = \theta_2 [a(1-s)^{H-1/\alpha} - bs^{H-1/\alpha}]$. For fixed $s \in (0, 1)$ we obtain $e^{tH} v(s, t) \rightarrow -\theta_1 bs^{H-1/\alpha}$ as $t \rightarrow \infty$, and from the mean value theorem we get

$$\begin{aligned} e^{tH} \{|u(s) + v(s, t)|^\alpha - |u(s)|^\alpha\} &\xrightarrow{t \rightarrow \infty} \\ -\alpha \theta_1 bs^{H-1/\alpha} \left| \theta_2 [a(1-s)^{H-1/\alpha} - bs^{H-1/\alpha}] \right|^{\alpha-1} \text{sgn} \left\{ \theta_2 [a(1-s)^{H-1/\alpha} - bs^{H-1/\alpha}] \right\} \\ &=: M_\infty(s). \end{aligned}$$

Since for the appropriate constants m_1 and m_2 we get

$$\begin{aligned} \sup_{t>1} e^{tH} ||v(s, t) + u(s)|^\alpha - |v(s, t)|^\alpha - |u(s)|^\alpha| \\ \leq \sup_{t>1} e^{tH} (\alpha + 1) |v(s, t)|^\alpha + \sup_{t>1} e^{tH} \alpha |v(s, t)| |u(s)|^{\alpha-1} \leq m_1 (1-s)^{H\alpha-1} + m_2 s^{H\alpha-1}, \end{aligned}$$

which is integrable on $(0, 1)$, the dominated convergence theorem yields

$$I_2(t) \sim e^{-tH} \int_0^1 M_\infty(s) ds.$$

For the next component we have $I_3(t) = \int_1^{e^t} \{|d(s, t) + f(s)|^\alpha - |d(s, t)|^\alpha - |f(s)|^\alpha\} ds$ with $d(s, t) = e^{-tH} \theta_1 [a(e^t - s)^{H-1/\alpha} - bs^{H-1/\alpha}]$ and $f(s) = \theta_2 [(s-1)^{H-1/\alpha} - s^{H-1/\alpha}]$. For fixed $s \in (1, \infty)$ we get

$$e^{tH} d(s, t) \xrightarrow{t \rightarrow \infty} -\theta_1 bs^{H-1/\alpha}$$

and from the mean-value theorem

$$e^{tH} \{|d(s, t) + f(s)|^\alpha - |f(s)|^\alpha\} \xrightarrow{t \rightarrow \infty} -\alpha \theta_1 b s^{H-1/\alpha} \left| \theta_2 b [(s-1)^{H-1/\alpha} - s^{H-1/\alpha}] \right|^{\alpha-1} \operatorname{sgn}\{\theta_2 b\} =: N_\infty(s).$$

Note that $N_\infty(s)$ is integrable on $(1, \infty)$. Set

$$N(s, t) := |d(s, t) + f(s)|^\alpha - |d(s, t)|^\alpha - |f(s)|^\alpha.$$

Then, we have

$$I_3(t) = \int_1^2 N(s, t) ds + \int_2^{e^t} N(s, t) ds =: I_{31}(t) + I_{32}(t).$$

We will find the rate of convergence for every I_{3i} , $i = 1, 2$, separately. For $I_{31}(t)$, fix $s \in (1, 2)$, then we get $e^{tH} N(s, t) \rightarrow N_\infty(s)$ as $t \rightarrow \infty$. Additionally,

$$\begin{aligned} \sup_{t>2} e^{tH} |N(s, t)| &\leq \sup_{t>2} e^{tH} (\alpha + 1) |d(s, t)|^\alpha + \sup_{t>2} e^{tH} \alpha |d(s, t)| |f(s)|^{\alpha-1} \\ &\leq p_1 [(3-s)^{H\alpha-1} + s^{H\alpha-1}] + p_2 [(3-s)^{H-1/\alpha} + s^{H-1/\alpha}] [(s-1)^{(H-1/\alpha)(\alpha-1)} + s^{(H-1/\alpha)(\alpha-1)}], \end{aligned}$$

which is integrable on $(1, 2)$. Here p_1 and p_2 are the appropriate constants independent of s and t . Hence,

$$I_{31}(t) \sim e^{-tH} \int_1^2 N_\infty(s) ds.$$

For $I_{32}(t)$ we need more subtle estimations. In what follows we show that

$$\left| \int_2^\infty [e^{tH} N(s, t) \mathbf{1}_{(2, e^t)}(s) - N_\infty(s)] ds \right| \xrightarrow{t \rightarrow \infty} 0$$

Fix $\epsilon > 0$ appropriately small and put

$$\begin{aligned} &\int_2^\infty [e^{tH} N(s, t) \mathbf{1}_{(2, e^t)}(s) - N_\infty(s)] ds \\ &= \int_2^{e^{tH}} \dots ds + \int_{e^{tH}}^{e^t - e^{tH} - \epsilon} \dots ds + \int_{e^t - e^{tH} - \epsilon}^{e^t - \epsilon} \dots ds + \int_{e^t - \epsilon}^{e^t} \dots ds + \int_{e^t}^\infty \dots ds \\ &=: J_1(t) + J_2(t) + J_3(t) + J_4(t) + J_5(t). \end{aligned}$$

Let us begin with $J_1(t)$. For fixed $s \in (2, \infty)$ we obtain

$$e^{tH} N(s, t) \mathbf{1}_{(2, e^{tH})}(s) \xrightarrow{t \rightarrow \infty} N_\infty(s).$$

Additionally, we use the fact that $s \in (2, e^{tH})$, which implies that there exist t_0 such that for every $t > t_0$ and every $s \in (2, e^{tH})$ we have $e^t - s > s$. Hence

$$\begin{aligned}
& \sup_{t>t_0} e^{tH} |N(s, t)| \mathbf{1}_{(2, e^{tH})}(s) \\
& \leq \sup_{t>t_0} e^{tH} (\alpha + 1) |d(s, t)|^\alpha \mathbf{1}_{(2, e^{tH})}(s) + \sup_{t>t_0} e^{tH} \alpha |d(s, t)| |f(s)|^{\alpha-1} \mathbf{1}_{(2, e^{tH})}(s) \\
& \leq \sup_{t>t_0} q_1 e^{tH} e^{-t\alpha H} s^{H\alpha-1} \mathbf{1}_{(2, e^{tH})}(s) + \sup_{t>t_0} q_2 s^{H-1/\alpha} (s-1)^{(H-1/\alpha-1)(\alpha-1)} \mathbf{1}_{(2, e^{tH})}(s) \\
& \leq q_1 s^{H\alpha-\alpha} + q_2 s^{H-1/\alpha} (s-1)^{(H-1/\alpha-1)(\alpha-1)},
\end{aligned}$$

which is integrable on $(2, \infty)$. Here q_1 and q_2 are the appropriate constants independent of s and t . Hence, the dominated convergence theorem yields $J_1(t) \rightarrow 0$ as $t \rightarrow \infty$.

For the next component we have

$$|J_2(t)| \leq \int_{e^{tH}}^{e^t - e^{tH} - \epsilon} e^{tH} |N(s, t)| ds + \int_{e^{tH}}^{e^t - e^{tH} - \epsilon} |N_\infty(s)| ds =: J_{21}(t) + J_{22}(t).$$

Since $N_\infty(s)$ is integrable on $(1, \infty)$, we get $J_{22}(t) \rightarrow 0$ as $t \rightarrow \infty$. Furthermore,

$$\begin{aligned}
J_{21}(t) & \leq \int_{e^{tH}}^{e^t - e^{tH} - \epsilon} e^{tH} (\alpha + 1) |f(s)|^\alpha ds + \int_{e^{tH}}^{e^t - e^{tH} - \epsilon} e^{tH} \alpha |f(s)| |d(s, t)|^{\alpha-1} ds \\
& \leq u_1 e^{tH} \int_{e^{tH}}^{e^t - e^{tH} - \epsilon} (s-1)^{H\alpha-1-\alpha} ds \\
& \quad + u_2 e^{tH} e^{-tH(\alpha-1)} \int_{e^{tH}}^{e^t - e^{tH} - \epsilon} (s-1)^{H-1/\alpha-1} (e^t - s)^{(H-1/\alpha)(\alpha-1)} ds \\
& \quad + u_3 e^{tH} e^{-tH(\alpha-1)} \int_{e^{tH}}^{e^t - e^{tH} - \epsilon} (s-1)^{H-1/\alpha-1} s^{(H-1/\alpha)(\alpha-1)} ds \xrightarrow{t \rightarrow \infty} 0,
\end{aligned}$$

where $u_i, i = 1, 2, 3$, are the appropriate constants independent of s, t and ϵ . Hence, $|J_2(t)| \rightarrow 0$ as $t \rightarrow \infty$.

Next, for the third term we get

$$|J_3(t)| \leq \int_{e^t - e^{tH} - \epsilon}^{e^t - \epsilon} e^{tH} |N(s, t)| ds + \int_{e^t - e^{tH} - \epsilon}^{e^t - \epsilon} |N_\infty(s)| ds =: J_{31}(t) + J_{32}(t).$$

Since $N_\infty(s)$ is integrable on $(1, \infty)$, we obtain $J_{32}(t) \rightarrow 0$ as $t \rightarrow \infty$. Additionally,

$$\begin{aligned}
J_{31}(t) &\leq \int_{e^t - e^{tH - \epsilon}}^{e^t - \epsilon} e^{tH} (\alpha + 1) |f(s)|^\alpha ds + \int_{e^t - e^{tH - \epsilon}}^{e^t - \epsilon} e^{tH} \alpha |f(s)| |d(s, t)|^{\alpha-1} ds \\
&\leq w_1 e^{tH} \int_{e^t - e^{tH - \epsilon}}^{e^t - \epsilon} (s - 1)^{H\alpha - 1 - \alpha} ds \\
&\quad + w_2 e^{tH} e^{-tH(\alpha-1)} \int_{e^t - e^{tH - \epsilon}}^{e^t - \epsilon} (s - 1)^{H-1/\alpha-1} \epsilon^{(H-1/\alpha)(\alpha-1)} ds \\
&\quad + w_3 e^{tH} e^{-tH(\alpha-1)} \int_{e^t - e^{tH - \epsilon}}^{e^t - \epsilon} (s - 1)^{H-1/\alpha-1} s^{(H-1/\alpha)(\alpha-1)} ds \xrightarrow{t \rightarrow \infty} 0,
\end{aligned}$$

where w_i , $i = 1, 2, 3$, are the appropriate constants independent of s , t and ϵ . Therefore, $|J_3(t)| \rightarrow 0$ as $t \rightarrow \infty$.

For the fourth part we put

$$|J_4(t)| \leq \int_{e^t - \epsilon}^{e^t} e^{tH} |N(s, t)| ds + \int_{e^t - \epsilon}^{e^t} |N_\infty(s)| ds =: J_{41}(t) + J_{42}(t).$$

Since $N_\infty(s)$ is integrable on $(1, \infty)$, we get $J_{42}(t) \rightarrow 0$ as $t \rightarrow \infty$. Further,

$$\begin{aligned}
J_{41}(t) &\leq \int_{e^t - \epsilon}^{e^t} e^{tH} (\alpha + 1) |d(s, t)|^\alpha ds + \int_{e^t - \epsilon}^{e^t} e^{tH} \alpha |d(s, t)| |f(s)|^{\alpha-1} ds \\
&\leq z_1 e^{tH} e^{-t\alpha H} \int_{e^t - \epsilon}^{e^t} [(e^t - s)^{H\alpha-1} + s^{H\alpha-1}] ds \\
&\quad + z_2 \int_{e^t - \epsilon}^{e^t} [(e^t - s)^{H-1/\alpha} + s^{H-1/\alpha}] (s - 1)^{(H-1/\alpha-1)(\alpha-1)} ds \leq z_3 \epsilon^{H\alpha} + z_4 \epsilon + z_5 \epsilon^{H-1/\alpha+1},
\end{aligned}$$

where z_i , $i = 1, \dots, 5$, are the appropriate constants independent of s , t and ϵ . Thus, we obtain

$$\lim_{\epsilon \searrow 0} \lim_{t \rightarrow \infty} |J_4(t)| = 0.$$

The last component $J_5(t) = \int_{e^t}^\infty N_\infty(s) ds \rightarrow 0$ as $t \rightarrow \infty$, since $N_\infty(s)$ is integrable on $(1, \infty)$. Finally, combining the results for $J_i(t)$, $i = 1, \dots, 5$, we obtain

$$\left| \int_2^\infty [e^{tH} N(s, t) \mathbf{1}_{(2, e^t)}(s) - N_\infty(s)] ds \right| \xrightarrow{t \rightarrow \infty} 0.$$

Hence, $I_{32}(t) \sim e^{-tH} \int_2^\infty N_\infty(s) ds$, which implies

$$I_3(t) = I_{31}(t) + I_{32}(t) \sim e^{-tH} \int_1^\infty N_\infty(s) ds$$

as $t \rightarrow \infty$.

>From part (i) of the proof we have $I_4(t) = O(e^{-t(1-H)})$. Additionally, the assumption $\alpha(1-H) > 1$ implies that $H < 1/2$. Thus, the contribution of $I_4(t)$ is negligible.

Finally, putting together the results for $I_j(t)$, $j = 1, \dots, 4$, we get

$$I(\theta_1; \theta_2; t) \sim e^{-tH} \left\{ \int_0^\infty H_\infty(s) ds + \int_0^1 M_\infty(s) ds + \int_1^\infty N_\infty(s) ds \right\}$$

as $t \rightarrow \infty$, which ends the proof of part (ii).

(iii) We begin with showing the following key inequality

Lemma

For $r > 0$, $s > 0$ and $\alpha \in (1, 2]$

$$(a) \quad |r^\alpha + s^\alpha - |r - s|^\alpha| = r^\alpha + s^\alpha - |r - s|^\alpha \leq (\alpha + 1)rs^{\alpha-1}. \quad (4.20)$$

$$(b) \quad |r^\alpha + s^\alpha - |r + s|^\alpha| = (r + s)^\alpha - r^\alpha - s^\alpha \leq \alpha rs^{\alpha-1}. \quad (4.21)$$

Proof of the Lemma:

(a) Let $r \geq s$. Define $f_s(r) := r^\alpha + s^\alpha - |r - s|^\alpha - (\alpha + 1)rs^{\alpha-1}$. We will show that $f_s(r) \leq 0$. We have $f_s(0) = 0$ and

$$\begin{aligned} f'_s(r) &= \alpha r^{\alpha-1} - \alpha(r - s)^{\alpha-1} - (\alpha + 1)s^{\alpha-1} \\ &\leq \alpha s^{\alpha-1} - (\alpha + 1)s^{\alpha-1} \leq 0. \end{aligned}$$

Thus $f_s(r) \leq 0$.

Let $r < s$. Using the mean-value theorem we get

$$\begin{aligned} r^\alpha + s^\alpha - (s - r)^\alpha &\leq rs^{\alpha-1} + \alpha r \int_0^1 [(s - r) + ru]^{\alpha-1} du \\ &\leq rs^{\alpha-1} + \alpha rs^{\alpha-1} = (\alpha + 1)rs^{\alpha-1}, \end{aligned}$$

which proves (4.20).

(b) We put $h_s(r) = r^\alpha + s^\alpha + \alpha rs^{\alpha-1} - (r + s)^\alpha$.

Since $h_s(0) = 0$ and $h'_s(r) = \alpha r^{\alpha-1} + \alpha s^{\alpha-1} - \alpha(r + s)^{\alpha-1} \geq 0$,

we get $h_s(r) \geq 0$. ■

Now, using the above result we determine the rate of convergence for every $I_j(t)$, $j = 1, \dots, 4$, from (4.8).

For $I_1(t)$, after some standard calculations, we have

$$I_1(t) = e^{t\alpha H} |a|^\alpha \int_0^\infty \{ |\bar{p}(s, t) + \bar{q}(s, t)|^\alpha - |\bar{p}(s, t)|^\alpha - |\bar{q}(s, t)|^\alpha \} ds$$

with $\bar{p}(s, t) = e^{-tH} \theta_1[(s+1)^{H-1/\alpha} - s^{H-1/\alpha}]$ and $\bar{q}(s, t) = \theta_2[(s+e^{-t})^{H-1/\alpha} - s^{H-1/\alpha}]$.

For fixed $s \in (0, \infty)$ we get

$$e^t \bar{q}(s, t) \xrightarrow{t \rightarrow \infty} (H - 1/\alpha) \theta_2 s^{H-1/\alpha-1}$$

and

$$\begin{aligned} & e^{t+tH(\alpha-1)} \{ |\bar{p}(s, t) + \bar{q}(s, t)|^\alpha - |\bar{p}(s, t)|^\alpha \} \xrightarrow{t \rightarrow \infty} \\ & \alpha(H - 1/\alpha) \theta_2 |\theta_1|^{\alpha-1} \operatorname{sgn}\{\theta_1\} s^{H-1/\alpha-1} [(s+1)^{H-1/\alpha} - s^{H-1/\alpha}]^{\alpha-1} =: P_\infty(s). \end{aligned}$$

Note that $P_\infty(s)$ is integrable on $(0, \infty)$. To apply the dominated convergence theorem, we need to use the inequalities derived in the previously proved lemma. Therefore, we use the fact that the sign of $\bar{p}(s, t)$ and $\bar{q}(s, t)$ is determined only by θ_1 and θ_2 , respectively. First, for $\theta_1 > 0$ and $\theta_2 > 0$ we see that $\bar{p}(s, t) > 0$ and $\bar{q}(s, t) > 0$. Thus, using (4.21) we obtain

$$\begin{aligned} & \sup_{t>2} e^{t+tH(\alpha-1)} ||\bar{p}(s, t) + \bar{q}(s, t)|^\alpha - |\bar{p}(s, t)|^\alpha - |\bar{q}(s, t)|^\alpha| \\ & \leq \sup_{t>2} e^{t+tH(\alpha-1)} \alpha \bar{q}(s, t) \bar{p}(s, t)^{\alpha-1} \\ & \leq c_1 s^{H-1/\alpha-1} [(s+1)^{H-1/\alpha} - s^{H-1/\alpha}]^{\alpha-1}, \end{aligned}$$

which is integrable on $(0, \infty)$. Here c_1 is the appropriate constant independent of s and t . Using (4.20) and (4.21) one can proceed with analogous estimations for any other possible sign of θ_1 and θ_2 . Therefore, we get

$$I_1(t) \sim e^{-t(1-H)} |a|^\alpha \int_0^\infty P_\infty(s) ds$$

as $t \rightarrow \infty$.

For the next term we have $I_2(t) = \int_0^1 \{ |v(s, t) + u(s)|^\alpha - |v(s, t)|^\alpha - |u(s)|^\alpha \} ds$, with $v(s, t) = e^{-tH} \theta_1 [a(e^t - s)^{H-1/\alpha} - bs^{H-1/\alpha}]$ and $u(s) = \theta_2 [a(1-s)^{H-1/\alpha} - bs^{H-1/\alpha}]$. Additionally, for fixed $s \in (0, 1)$, we obtain $e^{t/\alpha} v(s, t) \rightarrow a\theta_1$ as $t \rightarrow \infty$, and by the mean value theorem

$$\begin{aligned} & e^{t/\alpha} \{ |u(s) + v(s, t)|^\alpha - |u(s)|^\alpha \} \xrightarrow{t \rightarrow \infty} \\ & \alpha a \theta_1 |\theta_2|^{\alpha-1} |a(1-s)^{H-1/\alpha} - bs^{H-1/\alpha}|^{\alpha-1} \operatorname{sgn}\{\theta_2 [a(1-s)^{H-1/\alpha} - bs^{H-1/\alpha}]\}. \end{aligned}$$

Note that the limit function is integrable on $(0, 1)$. Now, applying the dominated convergence theorem in a standard manner, we see that $I_2(t) = O(e^{-t/\alpha})$, thus its contribution is negligible, since $H > 1/\alpha$ and $1 < \alpha \leq 2$ imply $1/\alpha > 1-1/\alpha > 1-H$.

For the next component, after some standard calculations, we have $I_3(t) = e^{t\alpha H} \int_{e^{-t}}^1 \{ |\bar{w}(s, t) + \bar{z}(s, t)|^\alpha - |\bar{w}(s, t)|^\alpha - |\bar{z}(s, t)|^\alpha \} ds$, where

$$\bar{w}(s, t) = \theta_1 e^{-tH} [a(1-s)^{H-1/\alpha} - bs^{H-1/\alpha}]$$

and

$$\bar{z}(s, t) = \theta_2 b [(s - e^{-t})^{H-1/\alpha} - s^{H-1/\alpha}].$$

For fixed $s \in (0, 1)$, we get from the mean value theorem $e^t \bar{z}(s, t) \rightarrow -\theta_2 b (H - 1/\alpha) s^{H-1/\alpha-1}$ as $t \rightarrow \infty$, and by using the mean value theorem again,

$$\begin{aligned} & e^{t+tH(\alpha-1)} \{ |\bar{w}(s, t) + \bar{z}(s, t)|^\alpha - |\bar{w}(s, t)|^\alpha \} \xrightarrow{t \rightarrow \infty} \\ & -\alpha |\theta_1|^{\alpha-1} \theta_2 b (H - 1/\alpha) s^{H-1/\alpha-1} |a(1-s)^{H-1/\alpha} - bs^{H-1/\alpha}|^{\alpha-1} \times \\ & \operatorname{sgn}\{\theta_1 [a(1-s)^{H-1/\alpha} - bs^{H-1/\alpha}]\} \\ & =: Q_\infty(s). \end{aligned}$$

Note that $Q_\infty(s)$ is integrable on $(0, 1)$. Put $Q(s, t) := |\overline{w}(s, t) + \overline{z}(s, t)|^\alpha - |\overline{w}(s, t)|^\alpha - |\overline{z}(s, t)|^\alpha$. In what follows, we will show that

$$\left| \int_0^1 [e^{t+tH(\alpha-1)} Q(s, t) \mathbf{1}_{(e^{-t}, 1)}(s) - Q_\infty(s)] ds \right| \xrightarrow{t \rightarrow \infty} 0.$$

Fix $\epsilon > 0$ appropriately small and set

$$\begin{aligned} \int_0^1 [e^{t+tH(\alpha-1)} Q(s, t) \mathbf{1}_{(e^{-t}, 1)}(s) - Q_\infty(s)] ds &= \int_0^{e^{-t}} \dots ds + \int_{e^{-t}}^{e^{-t}+\epsilon} \dots ds + \int_{e^{-t}+\epsilon}^1 \dots ds \\ &=: J_1(t) + J_2(t) + J_3(t). \end{aligned}$$

We immediately obtain

$$|J_1(t)| \xrightarrow{t \rightarrow \infty} 0,$$

since $Q_\infty(s)$ is integrable on $(0, 1)$. Next, we have

$$|J_2(t)| \leq \int_{e^{-t}}^{e^{-t}+\epsilon} e^{t+tH(\alpha-1)} |Q(s, t)| ds + \int_{e^{-t}}^{e^{-t}+\epsilon} |Q_\infty(s)| ds =: J_{21}(t) + J_{22}(t).$$

To find the appropriate upper bound for $J_{21}(t)$, we need to use inequalities (4.20) and (4.21). Therefore, we have to determine the signs of $\overline{w}(s, t)$ and $\overline{z}(s, t)$, which in turn depend on the parameters θ_1 , θ_2 , a and b .

The first case is $\theta_1 > 0$, $\theta_2 > 0$, $a > 0$ and $b > 0$. Then, we have $\overline{z}(s, t) \leq 0$ for every s , and

$$\begin{aligned} \overline{w}(s, t) &\geq 0 \quad \text{for } s \leq \frac{1}{1 + (b/a)^{1/\alpha-H}}, \\ \overline{w}(s, t) &\leq 0 \quad \text{for } s \geq \frac{1}{1 + (b/a)^{1/\alpha-H}}. \end{aligned}$$

Thus, using (4.20) and (4.21), we obtain

$$\begin{aligned} J_{21}(t) &= \int_{e^{-t}}^{e^{-t}+\epsilon} e^{t+tH(\alpha-1)} |Q(s, t)| ds \\ &= \int_{e^{-t}}^{e^{-t}+\epsilon} e^{t+tH(\alpha-1)} |Q(s, t)| \mathbf{1}_{\left(0, \frac{1}{1+(b/a)^{1/\alpha-H}}\right]}(s) ds \\ &\quad + \int_{e^{-t}}^{e^{-t}+\epsilon} e^{t+tH(\alpha-1)} |Q(s, t)| \mathbf{1}_{\left(\frac{1}{1+(b/a)^{1/\alpha-H}}, 1\right)}(s) ds \\ &\leq \int_{e^{-t}}^{e^{-t}+\epsilon} e^{t+tH(\alpha-1)} (\alpha + 1) |\overline{z}(s, t)| |\overline{w}(s, t)|^{\alpha-1} \\ &\quad + \int_{e^{-t}}^{e^{-t}+\epsilon} e^{t+tH(\alpha-1)} \alpha |\overline{z}(s, t)| |\overline{w}(s, t)|^{\alpha-1} ds \\ &\leq d_1 \int_{e^{-t}}^{e^{-t}+\epsilon} (s - e^{-t})^{H-1/\alpha-1} ds \leq d_2 \epsilon^{H-1/\alpha}. \end{aligned} \tag{4.22}$$

Here d_1 and d_2 are the appropriate constants independent of ϵ , s and t . For any other possible signs of the parameters θ_1 , θ_2 , a and b , first we determine the sign of $\bar{w}(s, t)$ and $\bar{z}(s, t)$, and next we proceed with the analogous estimations and find the same upper bound $J_{21}(t) \leq d\epsilon^{H-1/\alpha}$ with the appropriate constant d . For $J_{22}(t)$, we get

$$J_{22}(t) \leq d_3 \int_{e^{-t}}^{e^{-t}+\epsilon} s^{H-1/\alpha-1} ds = \frac{d_3}{H-1/\alpha} [(e^{-t}+\epsilon)^{H-1/\alpha} - (e^{-t})^{H-1/\alpha}],$$

where d_3 is the appropriate constant independent of ϵ , s and t . Finally, we obtain

$$\lim_{\epsilon \searrow 0} \lim_{t \rightarrow \infty} J_{2i}(t) = 0,$$

$i = 1, 2$, which implies that $\lim_{\epsilon \searrow 0} \lim_{t \rightarrow \infty} |J_2(t)| = 0$.

Now, we proceed with $J_3(t)$. For fixed $s \in (0, 1)$ we have

$$e^{t+tH(\alpha-1)} Q(s, t) \mathbf{1}_{(e^{-t}+\epsilon, 1)}(s) \xrightarrow{t \rightarrow \infty} Q_\infty(s) \mathbf{1}_{(\epsilon, 1)}(s).$$

To apply the dominated convergence theorem, we use the same method, as for $J_2(t)$, and determine the sign of $\bar{w}(s, t)$ and $\bar{z}(s, t)$. First, for $\theta_1 > 0$, $\theta_2 > 0$, $a > 0$ and $b > 0$ we have

$$\begin{aligned} & \sup_{t>2} e^{t+tH(\alpha-1)} |Q(s, t)| \mathbf{1}_{(e^{-t}+\epsilon, 1)}(s) \\ & \leq \sup_{t>2} e^{t+tH(\alpha-1)} |Q(s, t)| \mathbf{1}_{(e^{-t}+\epsilon, 1)}(s) \mathbf{1}_{\left(0, \frac{1}{1+(b/a)^{1/\alpha-H}}\right]}(s) \\ & \quad + \sup_{t>2} e^{t+tH(\alpha-1)} |Q(s, t)| \mathbf{1}_{(e^{-t}+\epsilon, 1)}(s) \mathbf{1}_{\left(\frac{1}{1+(b/a)^{1/\alpha-H}}, 1\right)}(s) \\ & \leq \sup_{t>2} e^{t+tH(\alpha-1)} \mathbf{1}_{(e^{-t}+\epsilon, 1)}(s) (\alpha+1) |\bar{z}(s, t)| |\bar{w}(s, t)|^{\alpha-1} \\ & \quad + \sup_{t>2} e^{t+tH(\alpha-1)} \mathbf{1}_{(e^{-t}+\epsilon, 1)}(s) \alpha |\bar{z}(s, t)| |\bar{w}(s, t)|^{\alpha-1} \\ & \leq d_5 \sup_{t>2} \mathbf{1}_{(e^{-t}+\epsilon, 1)}(s) (s - e^{-t})^{H-1/\alpha-1} \leq d_5 \epsilon^{H-1/\alpha-1}, \end{aligned}$$

which is integrable on $(\epsilon, 1)$. Here d_5 is the appropriate constant independent of ϵ , s and t . We proceed with the analogous estimations for any other sign of the parameters θ_1 , θ_2 , a and b . Thus, the dominated convergence theorem yields $|J_3(t)| \rightarrow 0$ as $t \rightarrow \infty$. Now, combining the results for every $J_i(t)$, $i = 1, 2, 3$, we obtain

$$\left| \int_0^1 [e^{t+tH(\alpha-1)} Q(s, t) \mathbf{1}_{(e^{-t}, 1)}(s) - Q_\infty(s)] ds \right| \xrightarrow{t \rightarrow \infty} 0$$

and consequently

$$I_3(t) \sim e^{-t(1-H)} \int_0^1 Q_\infty(s) ds$$

as $t \rightarrow \infty$.

For the last term, after some standard calculations, we have $I_4(t) = e^{t\alpha H} \int_1^\infty \{|\bar{g}(s, t) + \bar{h}(s, t)|^\alpha - |\bar{g}(s, t)|^\alpha - |\bar{h}(s, t)|^\alpha\} ds$ with

$$\bar{g}(s, t) = \theta_1 b e^{-tH} [(s-1)^{H-1/\alpha} - s^{H-1/\alpha}]$$

and

$$\bar{h}(s, t) = \theta_2 b [(s - e^{-t})^{H-1/\alpha} - s^{H-1/\alpha}].$$

> From the mean-value theorem, we get for fixed $s \in (1, \infty)$

$$e^{tH} \bar{h}(s, t) \xrightarrow{t \rightarrow \infty} -\theta_2 b (H - 1/\alpha) s^{H-1/\alpha-1}$$

and

$$\begin{aligned} & e^{t+tH(\alpha-1)} [|\bar{g}(s, t) + \bar{h}(s, t)|^\alpha - |\bar{g}(s, t)|^\alpha] \xrightarrow{t \rightarrow \infty} \\ & \alpha \theta_2 b (H - 1/\alpha) \operatorname{sgn}\{\theta_1 b\} |\theta_1 b|^{\alpha-1} |(s-1)^{H-1/\alpha} - s^{H-1/\alpha}|^{\alpha-1} s^{H-1/\alpha-1} =: R_\infty(s). \end{aligned}$$

Note that $R_\infty(s)$ is integrable on $(1, \infty)$. Put $R(s, t) := |\bar{g}(s, t) + \bar{h}(s, t)|^\alpha - |\bar{g}(s, t)|^\alpha - |\bar{h}(s, t)|^\alpha$. Now, applying the dominated convergence theorem in a standard manner, we show that

$$e^{t+tH(\alpha-1)} \int_1^\infty R(s, t) ds \xrightarrow{t \rightarrow \infty} \int_1^\infty R_\infty(s) ds,$$

which implies that

$$I_4(t) \sim e^{-t(1-H)} \int_1^\infty R_\infty(s) ds.$$

as $t \rightarrow \infty$.

Finally, combining the results for $I_j(t)$, $j = 1, \dots, 4$, we obtain

$$I(\theta_1; \theta_2; t) \sim e^{-t(1-H)} \left\{ |a|^\alpha \int_0^\infty P_\infty(s) ds + \int_0^1 Q_\infty(s) ds + \int_1^\infty R_\infty(s) ds \right\}$$

as $t \rightarrow \infty$, which ends part (iii) of the theorem. ■

The above theorems imply the following result, which is similar to the one for the Gaussian O-U process $Y_1(t)$ discussed in Section 2.2

Corollary 5. *The fractional O-U α -stable process $Z_1(t)$ does not have long memory in the sense of (3.3).*

PROOF: From Theorems 6, 7 and 8 we see that $I(\theta_1; \theta_2; t)$ decays exponentially as $t \rightarrow \infty$. Since, for the codifference, we have $\tau(t) = -I(1; -1; t)$, thus

$$\sum_{t=0}^\infty |\tau(t)| < \infty. \quad \blacksquare$$

4.2 Type II fractional α -stable Ornstein-Uhlenbeck process

In this section we introduce the second generalization $\{Z_2(t), t \in \mathbb{R}\}$ of the standard O-U process. $Z_2(t)$ can be viewed as the α -stable extension of the Gaussian process $Y_2(t)$, see formula (2.10).

First, we define the *finite-memory fractional α -stable motion* $\{\tilde{L}_{\alpha,H}(t), t \geq 0\}$ as the following Riemann-Liouville fractional integral [28]

$$\tilde{L}_{\alpha,H}(t) = \frac{1}{\Gamma(H - 1/\alpha + 1)} \int_0^t (t-s)^{H-1/\alpha} dL_\alpha(s) \quad , t \geq 0 \quad , \quad (4.23)$$

where $H > 0$, $\alpha \in (0, 2]$, $\Gamma(\cdot)$ is the Gamma function and $L_\alpha(s)$ is the symmetric α -stable random measure with the Lebesgue measure as control measure. $\tilde{L}_{\alpha,H}(t)$ is the extension of (2.9) to the α -stable case. Observe that $(t-s)^{H-1/\alpha}$ is α -integrable on $(0, t)$ for every $t \geq 0$, thus $\tilde{L}_{\alpha,H}(t)$ is a well-defined α -stable process. Additionally, for $H = 1/\alpha$ we get the standard symmetric α -stable motion.

Since for every $a > 0$

$$\begin{aligned} \tilde{L}_{\alpha,H}(at) &= \frac{1}{\Gamma(H - 1/\alpha + 1)} \int_0^{at} (at-s)^{H-1/\alpha} dL_\alpha(s) \stackrel{d}{=} \\ &\stackrel{d}{=} \frac{1}{\Gamma(H - 1/\alpha + 1)} \int_0^t a^{H-1/\alpha+1/\alpha} (t-u)^{H-1/\alpha} dL_\alpha(u) = \\ &= a^H \tilde{L}_{\alpha,H}(t) \quad , \end{aligned}$$

thus, $\tilde{L}_{\alpha,H}(at)$ is H -self-similar, but unlike the fractional α -stable motion $L_{\alpha,H}(t)$ defined in (3.5), it does not have stationary increments.

Lamperti transformation [4] provides one-to-one correspondence between self-similar and stationary processes. The α -stable Ornstein-Uhlenbeck process can be derived through the Lamperti transformation of the α -stable motion. Following the same line, we define *Type II fractional α -stable O-U process* $\{Z_2(t), t \in \mathbb{R}\}$ as the Lamperti transformation of $\tilde{L}_{\alpha,H}(t)$, namely

Definition 6. *The process*

$$\begin{aligned} Z_2(t) &= e^{-tH} \tilde{L}_{\alpha,H}(e^t) \\ &= \frac{e^{-tH}}{\Gamma(H - 1/\alpha + 1)} \int_0^{e^t} (e^t - s)^{H-1/\alpha} dL_\alpha(s) \quad , t \in \mathbb{R} \quad . \end{aligned} \quad (4.24)$$

is called **Type II fractional α -stable Ornstein-Uhlenbeck process**.

Note that for $\alpha = 2$, the stationary process $Z_2(t)$ reduces to the Gaussian one defined in (2.10).

The next three theorems give precise formulas for the asymptotic behaviour of the generalized codifference $I(\theta_1; \theta_2; t)$ corresponding to $Z_2(t)$. Next, we show that similarly to the Gaussian case the process $Z_2(t)$ does not have long memory.

Theorem 9. *Let $0 < \alpha < 1$ and $H > 0$. Then the generalized codifference of $Z_2(t)$ satisfies*

$$I(\theta_1; \theta_2; t) \sim -c_{\alpha, H} \cdot |\theta_1|^\alpha \cdot e^{-t}$$

as $t \rightarrow \infty$, where

$$c_{\alpha, H} = \frac{1}{(\Gamma(H - 1/\alpha + 1))^\alpha} . \quad (4.25)$$

PROOF: In the proof, we will take advantage of the following inequality

$$||a|^\alpha - |b|^\alpha| \leq |a - b|^\alpha \quad (4.26)$$

valid for $0 < \alpha \leq 1$ and $a, b \in \mathbb{R}$.

Formula (4.5) and some standard calculations give the following

$$I(\theta_1; \theta_2; t) = c_{\alpha, H} \cdot \left(\int_0^1 I_1(t, s) ds + \int_0^1 I_2(t, s) ds \right) , \quad (4.27)$$

where

$$\begin{aligned} I_1(t, s) &= -|\theta_1|^\alpha e^{-tH\alpha} (e^t - s)^{H\alpha-1} , \\ I_2(t, s) &= \left| \theta_1 e^{-tH} (e^t - s)^{H-1/\alpha} + \theta_2 (1-s)^{H-1/\alpha} \right|^\alpha - |\theta_2|^\alpha (1-s)^{H\alpha-1} \end{aligned}$$

and $c_{\alpha, H}$ is given by (4.25).

Since for every $s \in (0, 1)$

$$e^t \cdot I_1(t, s) \longrightarrow -|\theta_1|^\alpha \text{ as } t \rightarrow \infty \quad (4.28)$$

and

$$\sup_{t>1} (e^t \cdot |I_1(t, s)|) \leq \begin{cases} |\theta_1|^\alpha & \text{if } H\alpha - 1 > 0 \\ |\theta_1|^\alpha (1-s)^{H\alpha-1} & \text{if } H\alpha - 1 < 0 \end{cases}$$

which belongs to $L^1(0, 1)$, from (4.28) and the dominated convergence theorem we get

$$\int_0^1 I_1(t, s) ds \sim -|\theta_1|^\alpha \cdot e^{-t} \text{ as } t \rightarrow \infty . \quad (4.29)$$

Similarly, for every $s \in (0, 1)$

$$e^t \cdot I_2(t, s) \longrightarrow 0 \text{ as } t \rightarrow \infty$$

and from inequality (4.26)

$$\sup_{t>1} (e^t \cdot |I_2(t, s)|) \leq \begin{cases} |\theta_1|^\alpha & \text{if } H\alpha - 1 > 0 \\ |\theta_1|^\alpha (1-s)^{H\alpha-1} & \text{if } H\alpha - 1 < 0 \end{cases}$$

which also belongs to $L^1(0, 1)$. Therefore, the dominated convergence theorem implies that

$$e^t \cdot \int_0^1 I_2(t, s) ds \longrightarrow 0 \text{ as } t \rightarrow \infty \quad (4.30)$$

and finally from (4.29) and (4.30) we get

$$I(\theta_1; \theta_2; t) \sim -c_{\alpha, H} \cdot |\theta_1|^\alpha \cdot e^{-t} \text{ as } t \rightarrow \infty ,$$

which completes the proof. ■

Theorem 10. *Let $\alpha = 1$, $H > 0$. Then the generalized codifference of $Z_2(t)$ satisfies*

(i) *if $\theta_1 \theta_2 > 0$ then $I(\theta_1; \theta_2; t) = 0$*

(ii) *if $\theta_1 \theta_2 < 0$ then $I(\theta_1; \theta_2; t) \sim -2 \cdot c_{1, H} \cdot |\theta_1| \cdot e^{-t}$ as $t \rightarrow \infty$,*

where $c_{1, H}$ is given in (4.25).

PROOF: For $\alpha = 1$ formula (4.27) yields

$$I(\theta_1; \theta_2; t) = c_{1, H} \cdot \left(\int_0^1 I_1(t, s) ds + \int_0^1 I_2(t, s) ds \right) ,$$

where

$$\begin{aligned} I_1(t, s) &= -|\theta_1| e^{-tH} (e^t - s)^{H-1} , \\ I_2(t, s) &= |\theta_1 e^{-tH} (e^t - s)^{H-1} + \theta_2 (1 - s)^{H-1}| - |\theta_2| (1 - s)^{H-1} . \end{aligned}$$

(i) If $\theta_1 \theta_2 > 0$ then clearly $I_1(t, s) + I_2(t, s) = 0$ and therefore $I(\theta_1; \theta_2; t) = 0$.

(ii) For $\theta_1 \theta_2 < 0$ we show in a similar manner as in Theorem 9 that

$$\int_0^1 I_1(t, s) ds \sim -|\theta_1| \cdot e^{-t} \text{ as } t \rightarrow \infty . \quad (4.31)$$

Further, for every $s \in (0, 1)$

$$e^t I_2(t, s) \longrightarrow -|\theta_1| \text{ as } t \rightarrow \infty .$$

Taking advantage of inequality (4.26) and the dominated convergence theorem we conclude that

$$\int_0^1 I_2(t, s) ds \sim -|\theta_1| \cdot e^{-t} \text{ as } t \rightarrow \infty \quad (4.32)$$

and finally from (4.31) and (4.32)

$$I(\theta_1; \theta_2; t) \sim -2 \cdot c_{1, H} \cdot |\theta_1| \cdot e^{-t} \text{ as } t \rightarrow \infty .$$

■

To prove the next theorem, we need the following lemma

Lemma 3. *If $1 < \alpha \leq 2$ then for every $a \geq 0$, $b \geq 0$ we have*

- (i) $|a - b|^\alpha \leq a^\alpha + b^\alpha$
- (ii) $||a - b|^\alpha - b^\alpha| \leq a^\alpha + \alpha ab^{\alpha-1}$
- (iii) $||a + b|^\alpha - b^\alpha| \leq a^\alpha + \alpha ab^{\alpha-1}$

PROOF:

- (i) We put $f_b(a) = |a - b|^\alpha - a^\alpha - b^\alpha$.
 - for $a \geq b$ we get $f_b(0) = 0$ and $f'_b(a) = \alpha(a - b)^{\alpha-1} - \alpha a^{\alpha-1} \leq 0$
 - for $a < b$ we get $f_b(0) = 0$ and $f'_b(a) = -\alpha(b - a)^{\alpha-1} - \alpha a^{\alpha-1} \leq 0$
 which gives $f_b(a) \leq 0$. ■
- (ii) >From (i) we have
 $|a - b|^\alpha - b^\alpha \leq a^\alpha \leq a^\alpha + \alpha ab^{\alpha-1}$.
 We put $g_b(a) = |a - b|^\alpha - b^\alpha + a^\alpha + \alpha ab^{\alpha-1}$.
 - for $a \geq b$ we get $g_b(0) = 0$ and
 $g'_b(a) = \alpha(a - b)^{\alpha-1} + \alpha a^{\alpha-1} + \alpha b^{\alpha-1} \geq 0$
 - for $a < b$ we get $g_b(0) = 0$ and
 $g'_b(a) = -\alpha(b - a)^{\alpha-1} + \alpha a^{\alpha-1} + \alpha b^{\alpha-1} \geq 0$
 which implies $g_b(a) \geq 0$. ■
- (iii) We put $h_b(a) = a^\alpha + b^\alpha + \alpha ab^{\alpha-1} - |a + b|^\alpha$.
 Since $h_b(0) = 0$ and $h'_b(a) = \alpha a^{\alpha-1} + \alpha b^{\alpha-1} - \alpha(a + b)^{\alpha-1} \geq 0$,
 we get $h_b(a) \geq 0$. ■

Theorem 11. *Let $1 < \alpha < 2$ and $H > 0$. Then the generalized codifference of $Z_2(t)$ satisfies*

$$I(\theta_1; \theta_2; t) \sim c_{\alpha, H} \cdot d_{\alpha, H} \cdot \theta_1 \frac{|\theta_2|^\alpha}{\theta_2} \cdot e^{-t/\alpha}$$

as $t \rightarrow \infty$, where

$$d_{\alpha, H} = \frac{\alpha}{H(\alpha - 1) + 1/\alpha} ,$$

and $c_{\alpha, H}$ is given by (4.25).

PROOF: >From (4.27) and for $1 < \alpha < 2$

$$I(\theta_1; \theta_2; t) = c_{\alpha, H} \cdot \left(\int_0^1 I_1(t, s) ds + \int_0^1 I_2(t, s) ds \right) . \quad (4.33)$$

For every $s \in (0, 1)$ we have

$$e^{t/\alpha} \cdot I_1(t, s) \longrightarrow 0 \text{ as } t \rightarrow \infty$$

and following the same line as in Theorem 9, we obtain

$$e^{t/\alpha} \int_0^1 I_1(t, s) ds \longrightarrow 0 \text{ as } t \rightarrow \infty . \quad (4.34)$$

Further, for $s \in (0, 1)$

$$e^{t/\alpha} I_2(t, s) \longrightarrow \alpha \cdot \theta_1 \cdot \frac{|\theta_2|^\alpha}{\theta_2} \cdot (1-s)^{(H-1/\alpha)(\alpha-1)} \text{ as } t \rightarrow \infty$$

and from Lemma 3 we get

$$\begin{aligned} \sup_{t>1} \{ e^{t/\alpha} \cdot |I_2(t, s)| \} &\leq \sup_{t>1} \{ |\theta_1|^\alpha (1 - se^{-t})^{H\alpha-1} + \alpha \cdot |\theta_1| |\theta_2|^{\alpha-1} \\ &\quad \times (1 - se^{-t})^{H-1/\alpha} (1-s)^{(H-1/\alpha)(\alpha-1)} \} \\ &\leq \begin{cases} |\theta_1|^\alpha + \alpha \cdot |\theta_1| |\theta_2|^{\alpha-1} & \text{if } H\alpha - 1 > 0 \\ (|\theta_1|^\alpha + \alpha \cdot |\theta_1| |\theta_2|^{\alpha-1}) (1-s)^{H\alpha-1} & \text{if } H\alpha - 1 < 0 \end{cases} \end{aligned}$$

which belongs to $L^1(0, 1)$. Thus, the dominated convergence theorem yields

$$e^{t/\alpha} \cdot \int_0^1 I_2(t, s) ds \longrightarrow \alpha \cdot \theta_1 \cdot \frac{|\theta_2|^\alpha}{\theta_2} \cdot \int_0^1 (1-s)^{(H-1/\alpha)(\alpha-1)} ds \quad (4.35)$$

as $t \rightarrow \infty$. Finally from (4.33), (4.34) and (4.35) we get

$$I(\theta_1; \theta_2; t) \sim c_{\alpha, H} \cdot \frac{\alpha}{H(\alpha-1) + 1/\alpha} \cdot \theta_1 \frac{|\theta_2|^\alpha}{\theta_2} \cdot e^{-t/\alpha}$$

as $t \rightarrow \infty$, which completes the proof. ■

The asymptotic behaviour of the generalized codifference investigated in the above three theorems indicates that

Corollary 6. *The fractional O-U process $Z_2(t)$ does not have long-memory property in the sense of (3.3).*

PROOF: >From Theorems 9, 10 and 11 we get that $I(\theta_1; \theta_2; t)$ decays exponentially. Thus

$$\sum_{n=0}^{\infty} |\tau(n)| < \infty ,$$

which proves the statement. ■

4.3 Type III fractional α -stable Ornstein-Uhlenbeck process

In this section we employ the techniques originating from the fractional calculus to obtain in an elegant way the fractional α -stable O-U process, introduced in the recent paper by M.S. Taqqu and R.L. Wolpert [31] (Sec. 4.2.2). They define fractional α -stable O-U process in the following way: For $0 < \alpha \leq 2$ let $L_\alpha(t)$ be the standard symmetric α -stable random measure with Lebesgue control measure. For $\lambda > 0$ construct a series of processes indexed by κ via the recursive recipe

$$\begin{aligned} X_1(t) &\stackrel{\text{def}}{=} \sqrt{2\lambda} \int_{-\infty}^t e^{-\lambda(t-s)} L_\alpha(ds), \\ &\vdots \\ X_\kappa(t) &\stackrel{\text{def}}{=} \int_{-\infty}^t \lambda e^{-\lambda(t-s)} X_{\kappa-1}(s) ds, \end{aligned}$$

which by the Fubini's theorem for stochastic integrals gives

$$X_\kappa(t) = \frac{\sqrt{2\lambda} \lambda^{\kappa-1}}{\Gamma(\kappa)} \int_{-\infty}^t (t-s)^{\kappa-1} e^{-\lambda(t-s)} L_\alpha(ds). \quad (4.36)$$

For arbitrary $\kappa > 1 - 1/\alpha$ equation (4.36) is taken as the definition of the fractional α -stable O-U process. Note that for $\kappa = 1$ we get the standard α -stable O-U process

Let us now recall the definition of the Bessel fractional derivative. Samko et al. [28] introduce the modified Bessel operator $G_\lambda^\kappa : \mathbb{R}^\mathbb{R} \longrightarrow \mathbb{R}^\mathbb{R}$ via its Fourier transform as

$$\widehat{G_\lambda^\kappa f} \stackrel{\text{def}}{=} \frac{1}{(\lambda - i\omega)^\kappa} \widehat{f}(\omega), \quad \lambda > 0, \kappa > 0,$$

from which we get that

$$(G_\lambda^\kappa f)(t) = \frac{1}{\Gamma(\kappa)} \int_{-\infty}^t (t-s)^{\kappa-1} e^{-\lambda(t-s)} f(s) ds.$$

Here, we use the notation \widehat{h} for the Fourier transform of the function h . One can verify that for fixed λ we have $G_\lambda^\kappa G_\lambda^\beta f = G_\lambda^{\kappa+\beta} f$, so that the family of operators $\{G_\lambda^\kappa\}_{\kappa>0}$ forms a semigroup. The Bessel fractional derivative is now introduced as the operator inverse to G_λ^κ , i.e.

$$\left(\lambda I + \frac{d}{dt} \right)^\kappa f \stackrel{\text{def}}{=} (G_\lambda^\kappa)^{-1} f,$$

where I is the identity operator.

Now, we can introduce

Definition 7. *The stationary stochastic process $\{Z_3(t), t \in \mathbb{R}\}$ defined as the solution of the following fractional Langevin equation*

$$\left(\lambda I + \frac{d}{dt} \right)^\kappa Z_3(t) = l_\alpha(t). \quad (4.37)$$

*is called **Type III fractional α -stable O-U process**.*

Here $\lambda > 0$, $\kappa > 0$ and $l_\alpha(t)$ is the symmetric α -stable noise, i.e. it has symmetric α -stable marginal distributions, its probability distribution is translation invariant and $l_\alpha(s)$ and $l_\alpha(t)$ are independent for $s \neq t$.

Note that for $\kappa = 1$, equation (4.37) becomes the standard α -stable Langevin equation and its stationary solution is the O-U process.

To solve equation (4.37), we apply the modified Bessel operator G_λ^κ to both sides of the equation. Thus the solution $Z_3(t)$ has the form

$$Z_3(t) = \frac{1}{\Gamma(\kappa)} \int_{-\infty}^t (t-s)^{\kappa-1} e^{-\lambda(t-s)} L_\alpha(ds), \quad (4.38)$$

where $L_\alpha(s)$ is the symmetric α -stable random measure with Lebesgue control measure. Formally $L_\alpha(t) = \int_0^t l_\alpha(s) ds$. For $\kappa > 1 - 1/\alpha$ the kernel in (4.38) belongs to the Lebesgue space $L^\alpha((-\infty, t), ds)$ and the stochastic integral is well defined in the sense of convergence in probability. We have

Proposition 2. *Solution of the fractional Langevin equation (4.37) is equal (up to a constant) to the fractional O-U α -stable process (4.36).*

PROOF: Comparing formulas (4.36) with (4.38) gives the wanted result. ■

Thus the standard techniques developed in fractional calculus allow us to obtain the stochastic process $Z_3(t)$ in an elegant way. Let us note that for $\alpha = 2$, $Z_3(t)$ reduces to the short-memory Gaussian process $Y_3(t)$ defined by formula (2.12). In the next three theorems we explore the asymptotic dependence structure of $Z_3(t)$ and answer the question of the presence of long memory in this process.

Theorem 12. *Let $0 < \alpha < 1$, $\kappa > 0$ and $\lambda > 0$. Then the generalized codifference of $Z_3(t)$ satisfies*

$$I(\theta_1; \theta_2; t) \sim -c_{\alpha, \kappa} \cdot |\theta_1|^\alpha \cdot \frac{1}{\lambda \alpha} \cdot t^{\alpha(\kappa-1)} e^{-\lambda \alpha t}$$

as $t \rightarrow \infty$, where

$$c_{\alpha, \kappa} = \left(\frac{1}{\Gamma(\kappa)} \right)^\alpha. \quad (4.39)$$

PROOF: Formula (4.5) with some standard calculations yield

$$I(\theta_1; \theta_2; t) = c_{\alpha, \kappa} \cdot \left(\int_0^\infty I_1(t, s) ds + \int_0^\infty I_2(t, s) ds \right), \quad (4.40)$$

where

$$\begin{aligned} I_1(t, s) &= -|\theta_1|^\alpha (t+s)^{\alpha(\kappa-1)} e^{-\lambda \alpha (t+s)}, \\ I_2(t, s) &= \left| \theta_1 (t+s)^{\kappa-1} e^{-\lambda(t+s)} + \theta_2 s^{\kappa-1} e^{-\lambda s} \right|^\alpha - |\theta_2|^\alpha s^{\alpha(\kappa-1)} e^{-\lambda \alpha s} \end{aligned}$$

and $c_{\alpha, \kappa}$ is given by (4.39).

For every $s \in (0, \infty)$ we get

$$e^{\lambda \alpha t} \cdot t^{-\alpha(\kappa-1)} I_1(t, s) \longrightarrow -|\theta_1|^\alpha e^{-\lambda \alpha s} \text{ as } t \rightarrow \infty \quad (4.41)$$

and

$$\sup_{t>1} (|e^{\lambda \alpha t} \cdot t^{-\alpha(\kappa-1)} I_1(t, s)|) \leq \begin{cases} |\theta_1|^\alpha e^{-\lambda \alpha s} & \text{if } -1 < \alpha(\kappa-1) \leq 0 \\ |\theta_1|^\alpha (1+s)^{\alpha(\kappa-1)} e^{-\lambda \alpha s} & \text{if } \alpha(\kappa-1) > 0 \end{cases}$$

which belongs to $L^1((0, \infty), ds)$. Thus from the dominated convergence theorem

$$\begin{aligned} \int_0^\infty I_1(t, s) ds &\sim -|\theta_1|^\alpha \cdot e^{-\lambda \alpha t} \cdot t^{\alpha(\kappa-1)} \int_0^\infty e^{-\lambda \alpha s} ds \\ &= -|\theta_1|^\alpha \cdot e^{-\lambda \alpha t} \cdot t^{\alpha(\kappa-1)} \cdot \frac{1}{\lambda \alpha} \end{aligned} \quad (4.42)$$

as $t \rightarrow \infty$.

Further, for every $s \in (0, \infty)$, $e^{\lambda \alpha t} \cdot t^{-\alpha(\kappa-1)} \cdot I_2(t, s) \longrightarrow 0$ as $t \rightarrow \infty$, and since for $\alpha \in (0, 1]$, $||a|^\alpha - |b|^\alpha| \leq |a - b|^\alpha$, $a \in \mathbb{R}$, $b \in \mathbb{R}$, we have that $|I_2(t, s)| \leq |I_1(t, s)|$ and consequently

$$\sup_{t>1} (|e^{\lambda \alpha t} \cdot t^{-\alpha(\kappa-1)} I_2(t, s)|) \leq \begin{cases} |\theta_1|^\alpha e^{-\lambda \alpha s} & \text{if } -1 < \alpha(\kappa-1) \leq 0 \\ |\theta_1|^\alpha (1+s)^{\alpha(\kappa-1)} e^{-\lambda \alpha s} & \text{if } \alpha(\kappa-1) > 0 \end{cases}$$

which also belongs to $L^1((0, \infty), ds)$. Therefore, from the dominated convergence theorem we get

$$e^{\lambda \alpha t} \cdot t^{-\alpha(\kappa-1)} \cdot \int_0^\infty I_2(t, s) ds \longrightarrow 0 \text{ as } t \rightarrow \infty \quad (4.43)$$

and finally from (4.42) and (4.43) we conclude

$$I(\theta_1; \theta_2; t) \sim -c_{\alpha, \kappa} \cdot |\theta_1|^\alpha \cdot \frac{1}{\lambda \alpha} \cdot t^{\alpha(\kappa-1)} e^{-\lambda \alpha t} \text{ as } t \rightarrow \infty$$

■

Theorem 13. Let $\alpha = 1$, $\kappa > 0$ and $\lambda > 0$. Then the generalized codifference of $Z_3(t)$ satisfies

(i) if $\theta_1 \theta_2 > 0$ then $I(\theta_1; \theta_2; t) = 0$

(ii) if $\theta_1 \theta_2 < 0$ then

$$I(\theta_1; \theta_2; t) \sim -2 \cdot c_{1, \kappa} \cdot |\theta_1| \cdot \frac{1}{\lambda} \cdot t^{\kappa-1} e^{-\lambda t}$$

as $t \rightarrow \infty$, where $c_{1, \kappa}$ is given by (4.39).

PROOF: Equation (4.40) for $\alpha = 1$ gives

$$I(\theta_1; \theta_2; t) = c_{1,\kappa} \cdot \left(\int_0^\infty I_1(t, s) ds + \int_0^\infty I_2(t, s) ds \right),$$

where

$$\begin{aligned} I_1(t, s) &= -|\theta_1|(t+s)^{\kappa-1} e^{-\lambda(t+s)}, \\ I_2(t, s) &= \left| \theta_1(t+s)^{\kappa-1} e^{-\lambda(t+s)} + \theta_2 s^{\kappa-1} e^{-\lambda s} \right| - |\theta_2| s^{\kappa-1} e^{-\lambda s} \end{aligned}$$

(i) If $\theta_1 \theta_2 > 0$ then evidently $I_1(t, s) + I_2(t, s) = 0$ and therefore $r(\theta_1; \theta_2; t) = 0$.

(ii) For $\theta_1 \theta_2 < 0$ we have for every $s \in (0, \infty)$

$$e^{\lambda t} t^{-(\kappa-1)} (I_1(t, s) + I_2(t, s)) \longrightarrow -2|\theta_1| e^{-\lambda s} \text{ as } t \rightarrow \infty$$

and we show in a similar manner as in Theorem 12 that

$$I(\theta_1; \theta_2; t) \sim -2 \cdot c_{1,\kappa} \cdot |\theta_1| \cdot \frac{1}{\lambda} \cdot t^{\kappa-1} e^{-\lambda t}$$

as $t \rightarrow \infty$. ■

Theorem 14. Let $1 < \alpha < 2$, $\kappa > 1 - 1/\alpha$ and $\lambda > 0$. Then the generalized codifference of $Z_3(t)$ satisfies

$$I(\theta_1; \theta_2; t) \sim c_{\alpha,\kappa} \cdot d_\alpha(\lambda; \kappa) \cdot \theta_1 \cdot \frac{|\theta_2|^\alpha}{\theta_2} \cdot t^{\kappa-1} e^{-\lambda t}$$

as $t \rightarrow \infty$, where

$$d_\alpha(\lambda; \kappa) = \alpha \cdot \frac{\Gamma((\kappa-1)(\alpha-1)+1)}{(\lambda\alpha)^{(\kappa-1)(\alpha-1)+1}},$$

and $c_{\alpha,\kappa}$ is given by (4.39).

PROOF: From (4.40) we have

$$I(\theta_1; \theta_2; t) = c_{\alpha,\kappa} \cdot \left(\int_0^\infty I_1(t, s) ds + \int_0^\infty I_2(t, s) ds \right). \quad (4.44)$$

For every $s \in (0, \infty)$ we get $e^{\lambda t} t^{-(\kappa-1)} \cdot I_1(t, s) \longrightarrow 0$ as $t \rightarrow \infty$ and since

$$\sup_{t>1} (|e^{\lambda t} \cdot t^{-(\kappa-1)} I_1(t, s)|) \leq \begin{cases} c_1 \cdot |\theta_1|^\alpha e^{-\lambda \alpha s} & \text{if } \kappa - 1 \leq 0 \\ c_2 \cdot |\theta_1|^\alpha e^{-\lambda s} & \text{if } \kappa - 1 > 0 \end{cases}$$

(c_1 and c_2 are appropriate constants dependent only on parameters α , λ and κ), the dominated convergence theorem yields

$$e^{\lambda t} t^{-(\kappa-1)} \cdot \int_0^\infty I_1(t, s) ds \longrightarrow 0 \text{ as } t \rightarrow \infty.$$

Moreover, for $s \in (0, \infty)$

$$e^{\lambda t} t^{-(\kappa-1)} \cdot I_2(t, s) \longrightarrow \alpha \cdot \theta_1 \frac{|\theta_2|^\alpha}{\theta_2} e^{-\lambda \alpha s} s^{(\kappa-1)(\alpha-1)}$$

as $t \rightarrow \infty$. Taking advantage of the following inequalities

$$||a - b|^\alpha - b^\alpha| \leq a^\alpha + \alpha a b^{\alpha-1} \text{ and}$$

$$||a + b|^\alpha - b^\alpha| \leq a^\alpha + \alpha a b^{\alpha-1}$$

valid for $a \geq 0, b \geq 0$ and $\alpha \in (1, 2]$, we obtain

$$\begin{aligned} & \sup_{t>1} (|e^{\lambda t} \cdot t^{-(\kappa-1)} I_2(t, s)|) \leq \\ & \leq \begin{cases} c_1 \cdot |\theta_1|^\alpha e^{-\lambda \alpha s} + \alpha |\theta_1| |\theta_2|^{\alpha-1} e^{-\lambda \alpha s} s^{(\kappa-1)(\alpha-1)} & \text{if } \kappa - 1 \leq 0 \\ c_2 \cdot |\theta_1|^\alpha e^{-\lambda s} + \alpha |\theta_1| |\theta_2|^{\alpha-1} e^{-\lambda \alpha s} s^{(\kappa-1)(\alpha-1)} (1+s)^{\kappa-1} & \text{if } \kappa - 1 > 0 \end{cases} \end{aligned}$$

which belongs to $L^1((0, \infty), ds)$. Thus, the dominated convergence theorem yields

$$e^{\lambda t} t^{-(\kappa-1)} \cdot \int_0^\infty I_2(t, s) ds \longrightarrow \alpha \cdot \theta_1 \frac{|\theta_2|^\alpha}{\theta_2} \int_0^\infty e^{-\lambda \alpha s} s^{(\kappa-1)(\alpha-1)} ds$$

as $t \rightarrow \infty$. Since

$$\int_0^\infty e^{-\lambda \alpha s} s^{(\kappa-1)(\alpha-1)} ds = \frac{\Gamma((\kappa-1)(\alpha-1)+1)}{(\lambda \alpha)^{(\kappa-1)(\alpha-1)+1}}$$

we receive the following

$$I(\theta_1; \theta_2; t) \sim c_{\alpha, \kappa} \cdot d_\alpha(\lambda; \kappa) \cdot \theta_1 \cdot \frac{|\theta_2|^\alpha}{\theta_2} \cdot t^{\kappa-1} e^{-\lambda t}$$

as $t \rightarrow \infty$ and the proof is completed. ■

Corollary 7. *The fractional O-U α -stable process $Z_3(t)$ does not have long-memory property in the sense of (3.3).*

PROOF: Theorems 12, 13 and 14 imply that the codifference $\tau(t)$ decays exponentially. Therefore

$$\sum_{n=0}^\infty |\tau(n)| < \infty,$$

and the statement holds. ■

The above result shows that the lack of long memory observed in the Gaussian case (2.12) occurs also in the more general α -stable case for arbitrary $\alpha \in (0, 2)$.

4.4 Langevin equation with fractional α -stable noise

The next generalization $\{Z_4(t), t \in \mathbb{R}\}$ of the α -stable O-U process can be obtained by replacing the α -stable noise $l_\alpha(t)$ in Langevin equation (4.2) with the fractional α -stable noise $l_{\alpha, H}(t) = \frac{dL_{\alpha, H}(t)}{dt}$. Recall that $L_{\alpha, H}(t)$ is the fractional α -stable motion (3.5). Thus, the Langevin equation with fractional α -stable noise takes the form

$$\frac{dZ_4(t)}{dt} + \lambda Z_4(t) = l_{\alpha, H}(t). \quad (4.45)$$

Here, $0 < \alpha < 2$ and $0 < H < 1$.

As shown in [15], the unique stationary solution of the fractional Langevin equation (4.45) exist for $\alpha < 1$ and $H > 1/\alpha$. The solution takes the form

$$Z_4(t) = L_{\alpha,H}(t) - \lambda \int_{-\infty}^t e^{-\lambda(t-s)} L_{\alpha,H}(s) ds. \quad (4.46)$$

The authors of [15] prove the following result

Theorem. ([15]) *The codifference of $Z_4(t)$ satisfies*

$$\tau(t) \sim -k(\alpha, H)t^{\alpha(H-1)} \text{ as } t \rightarrow \infty,$$

where

$$\begin{aligned} k(\alpha, H) = \lambda^{-\alpha}(H - 1/\alpha)^\alpha & \left\{ |a|^\alpha \int_1^\infty \left(|(x-1)^{H-1/\alpha-1} - x^{H-1/\alpha-1}|^\alpha \right. \right. \\ & \left. - |(x-1)^{H-1/\alpha-1}|^\alpha - |x^{H-1/\alpha-1}|^\alpha \right) dx \\ & + \int_0^1 \left(|b(1-x)^{H-1/\alpha-1} - ax^{H-1/\alpha-1}|^\alpha \right. \\ & \left. - |b(1-x)^{H-1/\alpha-1}|^\alpha - |ax^{H-1/\alpha-1}|^\alpha \right) dx \\ & + |b|^\alpha \int_{-\infty}^0 \left(|(1-x)^{H-1/\alpha-1} - (-x)^{H-1/\alpha-1}|^\alpha \right. \\ & \left. - |(1-x)^{H-1/\alpha-1}|^\alpha - |(-x)^{H-1/\alpha-1}|^\alpha \right) dx \Bigg\}. \end{aligned}$$

The above theorem implies

Corollary 8. *The fractional O-U α -stable process $Z_4(t)$ has long memory in the sense of (3.3).*

PROOF: The power-law behaviour of the codifference implies that series (3.3) diverges. ■

As we can see, long memory present in the noise $l_\alpha^H(t)$ (see Corollary 1) transfers to the solution of fractional Langevin equation (4.45). Let us note that $Z_4(t)$ is the first fractional α -stable O-U process, between all the considered processes $Z_i(t)$, $i = 1, 2, 3, 4$, with long memory. This fact confirms the statement, that processes exhibiting long-range dependence are rather 'unusual'.

4.5 Link to FARIMA time series

In this section we define the continuous-time counterpart of the long-memory time series called FARIMA (fractional autoregressive integrated moving average). We prove

that the introduced stationary process has exactly the same dependence structure as FARIMA, and therefore, it is also a long-memory process.

The FARIMA discrete-time processes have found widespread acceptance as the mathematical models for various empirical time series with long memory ([2, 7] and references therein). We begin with recalling the definition. Let B be the shift operator defined by $BX(t) = X(t-1)$ and Δ be the difference operator i.e. $\Delta X(t) = X(t) - X(t-1) = (I - B)X(t)$. The FARIMA model is the generalization of the classical ARIMA(p, κ, q) model

$$\Phi(B)\Delta^\kappa X(t) = \Theta(B)\epsilon_t, \quad t \in \mathbb{N}.$$

Here Φ and Θ are the polynomials of degree p and q respectively, ϵ_n are assumed to be i.i.d symmetric α -stable random variables and κ is a non-negative integer. Now, for FARIMA(p, κ, q) the parameter κ is allowed to take also fractional values, either positive or negative. To avoid unnecessary complications, in our further discussion we set $p = q = 0$. Then the model is described by

$$\Delta^\kappa X(t) = \epsilon_t, \quad (4.47)$$

and consequently $X(t) = \Delta^{-\kappa}\epsilon_t$, where the operator $\Delta^{-\kappa} = (1 - B)^{-\kappa}$ for the fractional parameter κ is formally interpreted via the Taylor expansion of the function $(1 - z)^{-\kappa} = \sum_{j=0}^{\infty} b_j(-\kappa)z^j$. The coefficients in the series are

$$b_j(-\kappa) = \frac{\Gamma(j + \kappa)}{\Gamma(\kappa)\Gamma(j + 1)}. \quad (4.48)$$

Thus the formal definition of FARIMA($0, \kappa, 0$) process is the following

$$X(t) = \Delta^{-\kappa}\epsilon_t = (1 - B)^{-\kappa}\epsilon_t = \sum_{j=0}^{\infty} b_j(-\kappa)\epsilon_{t-j}, \quad t \in \mathbb{Z}. \quad (4.49)$$

$X(t)$ is a stationary moving average and the necessary condition for the series (4.49) to converge a.s. is $-\infty < \kappa < 1 - 1/\alpha$. In the Gaussian case, i.e. when $\alpha = 2$, the rate of decay of the covariance function $\text{Cov}(t) := E[X(t)X(0)] - E[X(t)]E[X(0)]$ for FARIMA model is $t^{2\kappa-1}$, [3], which shows that for $\kappa \geq 0$ we have $\sum_{n=0}^{\infty} |\text{Cov}(n)| = \infty$ and $X(t)$ is a process with long-range dependence. Additionally, the spectral density $f(\omega)$ (Fourier transform of $\text{Cov}(t)$) satisfies $f(\omega) \sim c|\omega|^{-2\kappa}$ as $\omega \rightarrow 0$. For $\alpha < 2$ the covariance doesn't exist and one has to employ other measures of dependence, appropriate for the stochastic processes with infinite second moment. In [12] authors determine the asymptotic behaviour of the codifference $\tau(t)$ for FARIMA(p, κ, q). They prove the following result

Theorem. ([12]) *Suppose $X(t) = \sum_{j=0}^{\infty} b_j(-\kappa)\epsilon_{t-j}$ is a FARIMA($0, \kappa, 0$) process with symmetric α -stable innovations ϵ_t , $t \in \mathbb{Z}$. Suppose $0 < \alpha \leq 2$ and κ is not an integer.*

(a) If either (i) $\alpha \leq 1$ or (ii) $\alpha > 1$ and $(\alpha - 1)(\kappa - 1) > -1$, then

$$\lim_{t \rightarrow \infty} \frac{\tau(t)}{t^{\alpha(\kappa-1)+1}} = \frac{1}{[\Gamma(\kappa)]^\alpha} \int_0^\infty g(x) dx,$$

where

$$g(x) = x^{(\kappa-1)\alpha} + (1+x)^{(\kappa-1)\alpha} - (x^{\kappa-1} - (1+x)^{\kappa-1})^\alpha.$$

(b) If $\alpha > 1$ and $(\alpha - 1)(\kappa - 1) < -1$, then

$$\lim_{t \rightarrow \infty} \frac{\tau(t)}{t^{\kappa-1}} = \frac{\alpha}{\Gamma(\kappa)} \sum_{j=0}^{\infty} b_j(-\kappa)^{<\alpha-1>}.$$

As a consequence, we obtain the following conclusion

Corollary 9. For $\kappa > 1 - 2/\alpha$ the FARIMA(0, κ , 0) process has long memory in the sense of (3.3).

PROOF: The condition $\kappa > 1 - 2/\alpha$ is equivalent to $\alpha(\kappa - 1) + 1 > -1$ and therefore for $\alpha \leq 1$ we have $\sum_{t=0}^{\infty} |\tau(t)| = \infty$. For $\alpha \in (1, 2]$ $\kappa > 1 - 2/\alpha$ implies $(\alpha - 1)(\kappa - 1) > -1$, thus $\tau(t) \sim ct^{\alpha(\kappa-1)+1}$ and $\sum_{t=0}^{\infty} |\tau(t)| = \infty$. ■

The arising question is, whether we can find a stationary α -stable process $Z(t)$ with continuous time t , which could be regarded as an appropriate counterpart of FARIMA(0, κ , 0) in the sense of the dependence structure. First, we proceed to the following heuristic considerations. From the Stirling's Formula for the Gamma function

$$\Gamma(z) \sim e^{-z} z^{z-1/2} \sqrt{2\pi}, \text{ as } z \rightarrow \infty,$$

we get the asymptotic behaviour of the coefficients (4.48)

$$b_j(-\kappa) = \frac{\Gamma(j + \kappa)}{\Gamma(\kappa)\Gamma(j + 1)} \sim \frac{j^{\kappa-1}}{\Gamma(\kappa)} \text{ as } j \rightarrow \infty.$$

Therefore, the process $X(t)$ from (4.49) can be considered as an approximated sum for stochastic integral

$$\begin{aligned} X(t) &= \sum_{j=0}^{\infty} b_j(-\kappa) \epsilon_{t-j} = \sum_{j=-\infty}^t b_{t-j}(-\kappa) \epsilon_j = \\ &= \sum_{j=-\infty}^t \int_{j-1}^j b_{t-j}(-\kappa) \mathbf{1}_{(j-1, j]}(s) L_\alpha(ds) \approx \\ &\approx \frac{1}{\Gamma(\kappa)} \int_{-\infty}^t (t-s)^{\kappa-1} L_\alpha(ds), \end{aligned}$$

where $L_\alpha(s)$ is the symmetric α -stable random measure with Lebesgue control measure. Thus, we have related the continuous-parameter moving average process

$$Z(t) := \frac{1}{\Gamma(\kappa)} \int_{-\infty}^t (t-s)^{\kappa-1} L_\alpha(ds), \quad t \in \mathbb{R}, \quad (4.50)$$

to the FARIMA process.

On the other hand, the equation (4.47) can be replaced by its continuous-time counterpart, namely

$$\frac{d^\kappa}{dt^\kappa} Z(t) = l_\alpha(t), \quad (4.51)$$

where the difference operator Δ is replaced by the fractional derivative operator of the Riemann-Liouville type $\frac{d^\kappa}{dt^\kappa}$ (see [28]) and the sequence of i.i.d. variables ϵ_t is replaced by the symmetric α -stable noise $l_\alpha(t)$. Using the standard Fourier-Laplace Transform techniques we get the following stationary solution of fractional differential equation (4.51)

$$Z(t) := \frac{1}{\Gamma(\kappa)} \int_{-\infty}^t (t-s)^{\kappa-1} L_\alpha(ds),$$

where $L_\alpha(s)$ is the symmetric α -stable random measure with Lebesgue control measure. It is astonishing that the result is exactly the same as the one derived in (4.50), which suggest that the process $Z(t)$ can be regarded as the "proper" continuous-time version of FARIMA(0, κ , 0). However, the main problem in this case is that the kernel function in the representation (4.50) doesn't belong to $L^\alpha(\mathbb{R}, ds)$. Therefore $Z(t)$ is not well defined. Below we present three different ways how to avoid this difficulty.

4.5.1 Fractional Langevin equation

The first possibility is to replace the equation (4.51) by the introduced in Sec.4.3 fractional Langevin equation (4.37)

$$\left(\lambda I + \frac{d}{dt} \right)^\kappa Z_3(t) = l_\alpha(t),$$

and let the parameter $\lambda \searrow 0$. Since the solution of the above equation is given by

$$Z_3(t) = \frac{1}{\Gamma(\kappa)} \int_{-\infty}^t (t-s)^{\kappa-1} e^{-\lambda(t-s)} L_\alpha(ds),$$

the kernel function belongs to $L^\alpha(\mathbb{R}, ds)$ only for $\kappa > 1 - 1/\alpha$. Let us remind that for FARIMA processes exactly the opposite condition for κ is required, which undoubtedly causes some difficulties while comparing the properties of $Z_3(t)$ and FARIMA. However, if $1 \leq \alpha < 2$ and $\kappa > 1 - 1/\alpha$, then from the results in Section 4.3 we see that the codifference of the appropriately re-scaled $Z_3(t)$ satisfies

$$\tau(t) \sim e^{-\lambda t} t^{\kappa-1}$$

as $t \rightarrow \infty$. Thus, for small values of λ the asymptotic dependence structure of $Z_3(t)$ is similar, at least in some regions, to the dependence structure of FARIMA, (see Theorem 4.5 (b)). For all these reasons, $Z_3(t)$ may only serve as a continuous-time model related to FARIMA.

4.5.2 Fractional α -stable noise

The second idea is to introduce an increment process of $Z(t)$. We define formally

$$\widehat{Z}(t) := Z(t+1) - Z(t) = \frac{1}{\Gamma(\kappa)} \int_{\mathbb{R}} [(t+1-s)_+^{\kappa-1} - (t-s)_+^{\kappa-1}] L_{\alpha}(ds), \quad (4.52)$$

where $a_+ = \max\{a, 0\}$. Now, we see that for $\kappa \in (1 - 1/\alpha, 2 - 1/\alpha)$ the kernel function in (4.52) is α -integrable, since it behaves like $s^{\kappa-1}$ as $s \rightarrow 0$ and like $s^{\kappa-2}$ as $s \rightarrow \infty$. Thus, the definition of $\widehat{Z}(t)$ is correct. Let us emphasize that $\widehat{Z}(t)$ should be considered as an approximation of the increments of the FARIMA(0, κ , 0) model, and not as the process related directly to FARIMA. Putting $\kappa - 1 = H - 1/\alpha$ we see that $\widehat{Z}(t)$ is a version of the well-known fractional α -stable noise $l_{\alpha,H}(t)$ defined in (3.6). The process $l_{\alpha,H}(t)$ is a classical example of a long-memory α -stable process. It is defined as the increment process of the H -self-similar fractional α -stable motion $L_{\alpha,H}(t)$ (i.e. $l_{\alpha,H}(t) = L_{\alpha,H}(t+1) - L_{\alpha,H}(t)$). Therefore, we come to conclusion that the process $\widehat{Z}(t)$, which is regarded as an approximation of the increments of FARIMA(0, κ , 0), is a version of $l_{\alpha,H}(t)$. Thus, we obtain a link between two significant long memory processes, namely the FARIMA model and the stationary linear fractional stable noise, which confirms that in both cases the property of long-range dependence has the same origin. Additionally, we see that $l_{\alpha,H}(t)$ is related directly to the increments of FARIMA and that the actual relationship between the parameters of both models is

$$\kappa - 1 = H - 1/\alpha.$$

4.5.3 Continuous-time FARIMA process

The third possibility is to perturb the solution $Z(t)$ of the FARIMA-type equation (4.51) in order to get rid of the possible divergence of the integral at the origin. Note that in the first considered case, application of the fractional Langevin equation allowed us to avoid divergence of the integral in $-\infty$. Now, we introduce

Definition 8. *Let $\epsilon > 0$. Then the process*

$$Z_5(t) = \frac{1}{\Gamma(\kappa)} \int_{-\infty}^t (t-s+\epsilon)^{\kappa-1} L_{\alpha}(ds), \quad t \in \mathbb{R}, \quad (4.53)$$

*is called the **Continuous-time FARIMA process**.*

$Z_5(t)$ is a stationary moving average process. It is well defined for $\kappa < 1 - 1/\alpha$, which in contrast with first two studied cases, agrees exactly with the permissible range of the parameter κ for FARIMA. Therefore in this case it is possible to compare asymptotic properties of $Z_5(t)$ and FARIMA for the same κ . Thus, the primary task is to determine the asymptotic behaviour of the codifference $\tau(t)$ for $Z_5(t)$. We prove the following theorem

Theorem 15. Let $0 < \alpha \leq 2$ and $-\infty < \kappa < 1 - 1/\alpha$. Then the codifference $\tau(t)$ of $Z_5(t)$ process satisfies

(a) If either (i) $\alpha \leq 1$ or (ii) $\alpha > 1$ and $(\alpha - 1)(\kappa - 1) > -1$, then

$$\lim_{t \rightarrow \infty} \frac{\tau(t)}{t^{\alpha(\kappa-1)+1}} = \frac{1}{[\Gamma(\kappa)]^\alpha} \int_0^\infty g(x) dx,$$

where

$$g(x) = x^{(\kappa-1)\alpha} + (1+x)^{(\kappa-1)\alpha} - (x^{\kappa-1} - (1+x)^{\kappa-1})^\alpha.$$

(b) If $\alpha > 1$ and $(\alpha - 1)(\kappa - 1) < -1$, then

$$\lim_{t \rightarrow \infty} \frac{\tau(t)}{t^{\kappa-1}} = \frac{\alpha}{\Gamma(\kappa)} \int_0^\infty h(x) dx,$$

where

$$h(x) = \frac{(x + \epsilon)^{(\kappa-1)(\alpha-1)}}{[\Gamma(\kappa)]^{\alpha-1}}.$$

PROOF: We begin with part (a). Since

$$\begin{aligned} \tau(t) &= \ln E[\exp\{i(Z_5(t) - Z_5(0))\}] \\ &\quad - \ln E[\exp\{iZ_5(t)\}] - \ln E[\exp\{-iZ_5(0)\}], \end{aligned}$$

formula (4.5) with some standard calculations give

$$\begin{aligned} \tau(t) &= \frac{1}{[\Gamma(\kappa)]^\alpha} \int_0^\infty [(x + \epsilon)^{(\kappa-1)\alpha} + (t + x + \epsilon)^{(\kappa-1)\alpha} \\ &\quad - ((x + \epsilon)^{\kappa-1} - (t + x + \epsilon)^{\kappa-1})^\alpha] dx. \end{aligned} \quad (4.54)$$

After the change of variables $x \rightarrow tx$ we get

$$\tau(t) = \frac{t^{\alpha(\kappa-1)+1}}{[\Gamma(\kappa)]^\alpha} \int_0^\infty [|a_t(x)|^\alpha + |b_t(x)|^\alpha - |a_t(x) - b_t(x)|^\alpha] dx,$$

where $a_t(x) = (x + \epsilon/t)^{\kappa-1}$ and $b_t(x) = (1 + x + \epsilon/t)^{\kappa-1}$. Thus for fixed $x \in (0, \infty)$ we have

$$\begin{aligned} a_t(x) &\longrightarrow x^{\kappa-1} \text{ as } t \rightarrow \infty \text{ and} \\ b_t(x) &\longrightarrow (1+x)^{\kappa-1} \text{ as } t \rightarrow \infty. \end{aligned}$$

To apply the dominated convergence theorem, we need the following inequality [15]: For $r, s \in \mathbb{R}$

$$||r + s|^\alpha - |r|^\alpha - |s|^\alpha| \leq \begin{cases} 2|r|^\alpha & \text{if } 0 < \alpha \leq 1 \\ (\alpha + 1)|r|^\alpha + \alpha|r||s|^{\alpha-1} & \text{if } 1 < \alpha \leq 2. \end{cases}$$

Using the above result we obtain

$$\begin{aligned} & \sup_{t>1} ||a_t(x)|^\alpha + |b_t(x)|^\alpha - |a_t(x) - b_t(x)|^\alpha| \leq \\ & \leq \begin{cases} 2|1+x|^{\alpha(\kappa-1)} & \text{if } 0 < \alpha \leq 1 \\ (\alpha+1)|1+x|^{\alpha(\kappa-1)} + \alpha|1+x|^{\kappa-1}|x|^{(\alpha-1)(\kappa-1)} & \text{if } 1 < \alpha \leq 2, \end{cases} \end{aligned}$$

which in both cases belongs to $L^1((0, \infty), ds)$ (note that in the second case we assumed $(\alpha-1)(\kappa-1) > -1$). Thus the dominated convergence theorem yields

$$\lim_{t \rightarrow \infty} \frac{\tau(t)}{t^{\alpha(\kappa-1)+1}} = \frac{1}{[\Gamma(\kappa)]^\alpha} \int_0^\infty g(x) dx,$$

where $g(x) = x^{(\kappa-1)\alpha} + (1+x)^{(\kappa-1)\alpha} - (x^{\kappa-1} + (1+x)^{\kappa-1})^\alpha$.

We pass on to part (b) of the theorem. >From (4.54) we get that

$$\tau(t) = \frac{1}{[\Gamma(\kappa)]^\alpha} \int_0^\infty [(p(x))^\alpha + (q_t(x))^\alpha - (p(x) - q_t(x))^\alpha] dx,$$

where $p(x) = (x + \epsilon)^{\kappa-1}$ and $q_t(x) = (t + x + \epsilon)^{\kappa-1}$. Note that for fixed $x \in (0, \infty)$ we have $q_t(x) \sim t^{\kappa-1}$ as $t \rightarrow \infty$. >From the first order mean-value theorem

$$f(r+s) - f(r) = s \int_0^1 f'(r+us) du,$$

where f is an appropriately smooth function, we obtain

$$[p(x)]^\alpha - [p(x) - q_t(x)]^\alpha = \alpha q_t(x) \int_0^1 [p(x) - u q_t(x)]^{\alpha-1} du,$$

and consequently $[p(x)]^\alpha - [p(x) - q_t(x)]^\alpha \sim \alpha t^{\kappa-1} [p(x)]^{\alpha-1}$ as $t \rightarrow \infty$. Moreover, $\frac{[q_t(x)]^\alpha}{t^{\kappa-1}} \rightarrow 0$ as $t \rightarrow \infty$, since $\alpha > 1$. Thus for fixed $x \in (0, \infty)$ we have

$$\frac{[p(x)]^\alpha + [q_t(x)]^\alpha - [p(x) - q_t(x)]^\alpha}{t^{\kappa-1}} \rightarrow \alpha [p(x)]^{\alpha-1} = \alpha (x + \epsilon)^{(\alpha-1)(\kappa-1)}$$

as $t \rightarrow \infty$. To apply the dominated convergence theorem, we need the following inequality:

For $r, s > 0$ and $\alpha \in (1, 2]$

$$r^\alpha + s^\alpha - |r-s|^\alpha \leq (\alpha+1)rs^{\alpha-1}. \quad (4.55)$$

Proof of the inequality:

(i) Let $r \geq s$. Define $f_s(r) := r^\alpha + s^\alpha - |r-s|^\alpha - (\alpha+1)rs^{\alpha-1}$. We will show that $f_s(r) \leq 0$. We have $f_s(0) = 0$ and

$$\begin{aligned} f'_s(r) &= \alpha r^{\alpha-1} - \alpha(r-s)^{\alpha-1} - (\alpha+1)s^{\alpha-1} \leq \\ &\leq \alpha s^{\alpha-1} - (\alpha+1)s^{\alpha-1} \leq 0. \end{aligned}$$

Thus $f_s(r) \leq 0$.

(ii) Let $r < s$. Using the mean-value theorem we get

$$\begin{aligned} r^\alpha + s^\alpha - (s-r)^\alpha &\leq rs^{\alpha-1} + \alpha r \int_0^1 [(s-r) + ru]^{\alpha-1} du \leq \\ &\leq rs^{\alpha-1} + \alpha rs^{\alpha-1} = (\alpha+1)rs^{\alpha-1}, \end{aligned}$$

which proves (4.55).

Now, using the above result we obtain

$$\begin{aligned} \sup_{t>1} \frac{|[p(x)]^\alpha + [q_t(x)]^\alpha - [p(x) - q_t(x)]^\alpha|}{t^{\kappa-1}} &\leq \sup_{t>1} \frac{(\alpha+1)q_t(x)[p(x)]^{\alpha-1}}{t^{\kappa-1}} \leq \\ &\leq (\alpha+1)[p(x)]^{\alpha-1} = (\alpha+1)(x+\epsilon)^{(\alpha-1)(\kappa-1)}, \end{aligned}$$

which for $(\alpha-1)(\kappa-1) < -1$ belongs to $L^1((0, \infty), ds)$. Finally, the dominated convergence theorem yields

$$\lim_{t \rightarrow \infty} \frac{\tau(t)}{t^{\kappa-1}} = \frac{\alpha}{\Gamma(\kappa)} \int_0^\infty h(x) dx,$$

where

$$h(x) = \frac{(x+\epsilon)^{(\kappa-1)(\alpha-1)}}{[\Gamma(\kappa)]^{\alpha-1}}.$$

■

We get the conclusion

Corollary. *For $\kappa > 1 - 2/\alpha$ the process $Z_5(t)$ has long memory in the sense of (3.3).*

PROOF: It's enough to repeat the arguments from the proof of Corollary 9, since the rate of convergence of $\tau(t)$ does not depend on ϵ . ■

The above results for the codifference of $Z_5(t)$ are actually identical with the ones for FARIMA $(0, \kappa, 0)$ process. The rate of convergence of $\tau(t)$ in both cases is exactly the same and does not depend on ϵ , which implies that both processes have long-memory property for the same range of parameter κ . The parameter ϵ only affects the constant in part (b) of the above theorem, whereas the constant in part (a) is identical for both processes. For these reasons we may consider $Z_5(t)$ as the **"proper" continuous-time counterpart of FARIMA(0, κ , 0)** in the sense of the dependence structure. Since the FARIMA models, due to their long-memory property, have found their widespread acceptance in modelling various empirical time series, the introduced process $Z_5(t)$, thanks to his identical asymptotic dependence structure and simplicity of the definition, can serve as a useful continuous-time process in all those fields, where the FARIMA models were required.

Chapter 5

Correlation Cascade and the dependence structure

In the previous chapter we have investigated the dependence structure of the fractional processes $Z_i(t)$, $i = 1, \dots, 5$ in the language of the codifference. Now, we concentrate on the alternative measure of dependence – correlation cascade $C_l(\cdot)$ introduced in (3.9). In what follows, we derive the precise formulas for the asymptotic behaviour of $C_l(0, t)$ corresponding to the discussed processes. We detect long memory in the sense of definition (3.17) and, using the results from Section 3.2, we show that the examined processes are mixing.

5.1 Type I fractional α -stable Ornstein-Uhlenbeck process

In this section we investigate the asymptotic behaviour of the correlation cascade $C_l(0, t)$ for the process $Z_1(t)$ defined by (4.4). We verify the presence of long memory in this process and show that $Z_1(t)$ is mixing.

Theorem 16. *Let $0 < \alpha < 2$, $0 < H < 1$, $a \geq 0$, $b \leq 0$ and $H - 1/\alpha > 0$. Then the correlation cascade of $Z_1(t)$ satisfies*

$$C_l(0, t) \sim C \cdot e^{-t \frac{\alpha(1-H)}{\alpha(1-H)+1}} \quad \text{as } t \rightarrow \infty. \quad (5.1)$$

Here C is the appropriate positive constant dependant only on the parameters α and H .

PROOF: We have

$$C_l(0, t) = \int_{-\infty}^{\infty} \min\{f(s, t); f(s, 0)\}^\alpha ds,$$

where

$$f(s, t) = e^{-tH} a \left[(e^t - s)_+^{H-1/\alpha} - (-s)_+^{H-1/\alpha} \right] + e^{-tH} b \left[(e^t - s)_-^{H-1/\alpha} - (-s)_-^{H-1/\alpha} \right].$$

Let us introduce the following decomposition

$$\begin{aligned} C_l(0, t) &= \int_{-\infty}^{\infty} \min\{f(s, t) ; f(s, 0)\}^{\alpha} ds \\ &= \int_{-\infty}^0 \dots ds + \int_0^1 \dots ds + \int_1^{e^t} \dots ds + \int_{e^t}^{\infty} \dots ds =: I_1(t) + I_2(t) + I_3(t) + I_4(t). \end{aligned} \quad (5.2)$$

In what follows, we estimate the rate of convergence of every $I_j(t)$, $j = 1, \dots, 4$, separately.

Let us begin with $I_3(t)$. We have

$$I_3(t) = \int_1^{e^t} \min\{e^{-tH}[a(e^t - s)^{H-1/\alpha} - bs^{H-1/\alpha}] ; b[(s-1)^{H-1/\alpha} - s^{H-1/\alpha}]\}^{\alpha} ds.$$

Set

$$k(t) := e^{\frac{t}{\alpha(1-H)+1}} \left(\frac{a-b}{(-b)(H-1/\alpha)} \right)^{\frac{1}{H-1/\alpha-1}}. \quad (5.3)$$

Then, we have $k(t) \rightarrow \infty$ and $\frac{e^t}{k(t)} \rightarrow \infty$ as $t \rightarrow \infty$. Additionally, for $s \in (1, k(t))$ we have

$$\begin{aligned} e^{-tH}[a(e^t - s)^{H-1/\alpha} - bs^{H-1/\alpha}] &\leq (a-b)e^{-t/\alpha} = (-b)(H-1/\alpha)k(t)^{H-1/\alpha-1} \\ &\leq (-b)(H-1/\alpha) \int_0^1 (s-u)^{H-1/\alpha-1} du = b[(s-1)^{H-1/\alpha} - s^{H-1/\alpha}]. \end{aligned}$$

Therefore, for $s \in (1, k(t))$ we obtain

$$\begin{aligned} &\min\{e^{-tH}[a(e^t - s)^{H-1/\alpha} - bs^{H-1/\alpha}] ; b[(s-1)^{H-1/\alpha} - s^{H-1/\alpha}]\} \\ &= e^{-tH}[a(e^t - s)^{H-1/\alpha} - bs^{H-1/\alpha}]. \end{aligned} \quad (5.4)$$

Next, we set

$$I_3(t) = \int_1^{e^t} \min\{\dots\}^{\alpha} ds = \int_1^{k(t)} \min\{\dots\}^{\alpha} ds + \int_{k(t)}^{e^t} \min\{\dots\}^{\alpha} ds =: I_{31}(t) + I_{32}(t).$$

For $I_{31}(t)$, using (5.4), we obtain

$$I_{31}(t) = \int_1^{k(t)} e^{-tH\alpha}[a(e^t - s)^{H-1/\alpha} - bs^{H-1/\alpha}]^{\alpha} ds,$$

and, after substituting $s \rightarrow k(t)s$, we get

$$I_{31}(t) = e^{-tH\alpha}k(t)^{H\alpha} \int_{\frac{1}{k(t)}}^1 g(s, t) ds,$$

with

$$g(s, t) = \left[a \left(\frac{e^t}{k(t)} - s \right)^{H-1/\alpha} - b s^{H-1/\alpha} \right]^\alpha.$$

Since for fixed $s \in (0, 1)$ we have

$$g(s, t) \sim a^\alpha \left(\frac{e^t}{k(t)} \right)^{H\alpha-1}$$

as $t \rightarrow \infty$, the dominated convergence theorem yields

$$I_{31}(t) \sim c_1 e^{-t} k(t) \sim c_2 e^{-t \frac{\alpha(1-H)}{\alpha(1-H)+1}} \quad (5.5)$$

as $t \rightarrow \infty$. Here c_1 and c_2 are the appropriate positive constants independent of t . For $I_{32}(t)$ we have

$$\begin{aligned} I_{32}(t) &\leq (-b)^\alpha \int_{k(t)}^\infty [s^{H-1/\alpha} - (s-1)^{H-1/\alpha}] ds \\ &= (-b)^\alpha k(t)^{H\alpha} \int_1^\infty [s^{H-1/\alpha} - (s-1/k(t))^{H-1/\alpha}]^\alpha ds. \end{aligned}$$

>From the mean-value theorem we obtain

$$s^{H-1/\alpha} - (s-1/k(t))^{H-1/\alpha} = (H-1/\alpha) \frac{1}{k(t)} \int_0^1 (s-u/k(t))^{H-1/\alpha-1} du.$$

Thus, for fixed $s \in (1, \infty)$ we get

$$\frac{[s^{H-1/\alpha} - (s-1/k(t))^{H-1/\alpha}]^\alpha}{k(t)^{-\alpha}} \longrightarrow (H-1/\alpha)^\alpha s^{H\alpha-1-\alpha}$$

as $t \rightarrow \infty$, which is integrable on $(1, \infty)$. Consequently, the dominated convergence theorem yields

$$\int_1^\infty [s^{H-1/\alpha} - (s-1/k(t))^{H-1/\alpha}]^\alpha ds \sim c_3 k(t)^{-\alpha}$$

where c_3 is the appropriate positive constant, and finally

$$I_{32}(t) = O(k(t)^{H\alpha-\alpha}) = O\left(e^{-t \frac{\alpha(1-H)}{\alpha(1-H)+1}}\right).$$

Combining the above result with (5.5) we obtain

$$I_3(t) \sim c_4 e^{-t \frac{\alpha(1-H)}{\alpha(1-H)+1}} \quad (5.6)$$

as $t \rightarrow \infty$, where c_4 is the appropriate positive constant independent of t .

Recall the decomposition (5.2). We continue our estimations with $I_1(t)$. We have

$$I_1(t) = \int_{-\infty}^0 \min\{h(s, t); h(s, 0)\}^\alpha ds,$$

where

$$h(s, t) = ae^{-tH}[(e^t - s)^{H-1/\alpha} - (-s)^{H-1/\alpha}].$$

After some standard calculations we obtain the decomposition

$$\begin{aligned} I_1(t) &= \int_0^\infty \min\{h(-s, t); h(-s, 0)\}^\alpha ds \\ &= \int_0^{k(t)} \min\{...\}^\alpha ds + \int_{k(t)}^\infty \min\{...\}^\alpha ds =: I_{11}(t) + I_{12}(t), \end{aligned}$$

where $k(t)$ is given by (5.3). Additionally, we have

$$\begin{aligned} I_{11}(t) &\leq \int_0^{k(t)} a^\alpha e^{-tH\alpha} [(e^t + s)^{H-1/\alpha} - s^{H-1/\alpha}]^\alpha ds, \\ I_{12}(t) &\leq \int_{k(t)}^\infty a^\alpha [(1 + s)^{H-1/\alpha} - s^{H-1/\alpha}]^\alpha ds, \end{aligned}$$

The two integrals on the right-hand sides of the above inequalities are $O\left(e^{-t\frac{\alpha(1-H)}{\alpha(1-H)+1}}\right)$ functions. The proof of this fact is analogous to the one carried out for $I_{31}(t)$ and $I_{32}(t)$. Therefore, we immediately obtain

$$I_1(t) = I_{11}(t) + I_{12}(t) = O\left(e^{-t\frac{\alpha(1-H)}{\alpha(1-H)+1}}\right) \quad (5.7)$$

as $t \rightarrow \infty$.

For the term $I_2(t)$ in decomposition (5.2) we have

$$I_2(t) \leq e^{-tH\alpha} \int_0^1 [a(e^t - s)^{H-1/\alpha} - bs^{H-1/\alpha}]^\alpha ds \leq e^{-tH\alpha} a^\alpha e^{t(H\alpha-1)} = a^\alpha e^{-t}.$$

Therefore, we obtain

$$I_2(t) = O(e^{-t}) \quad (5.8)$$

as $t \rightarrow \infty$. Thus, it decays faster than $I_3(t)$ and its contribution is negligible.

For the last term in decomposition (5.2) we get after some standard calculations

$$\begin{aligned} I_4(t) &\leq \int_{e^t}^\infty (-b)^\alpha [s^{H-1/\alpha} - (s-1)^{H-1/\alpha}]^\alpha ds \\ &= (-b)^\alpha e^{tH\alpha} \int_1^\infty [s^{H-1/\alpha} - (s - e^{-t})^{H-1/\alpha}]^\alpha ds. \end{aligned}$$

For fixed $s \in (1, \infty)$ we have

$$s^{H-1/\alpha} - (s - e^{-t})^{H-1/\alpha} = (H - 1/\alpha)e^{-t} \int_0^1 (s - ue^{-t})^{H-1/\alpha-1} du.$$

Therefore, from the dominated convergence theorem, we obtain

$$\int_1^\infty [s^{H-1/\alpha} - (s - e^{-t})^{H-1/\alpha}]^\alpha ds \sim c_5 e^{-t\alpha}$$

as $t \rightarrow \infty$. Here c_5 is the appropriate positive constant. Consequently

$$I_4(t) = O(e^{-t\alpha(1-H)}). \quad (5.9)$$

as $t \rightarrow \infty$. Since $\alpha(1-H) + 1 > 1$, we get that $I_4(t)$ decays faster than $I_3(t)$, thus, its contribution is negligible.

Finally, comparing the results in (5.6) – (5.9), we obtain

$$C_l(0, t) \sim C \cdot e^{-t \frac{\alpha(1-H)}{\alpha(1-H)+1}} \quad \text{as } t \rightarrow \infty,$$

where C is the appropriate positive constant dependant only on the parameters α and H . ■.

>From the above theorem we get the following conclusions

Corollary 10. *The process $Z_1(t)$ does not have long memory in the sense of (3.17).*

PROOF: Since the correlation cascade $C_l(0, t)$ decays exponentially, the series (3.17) converges. ■

Corollary 11. *The process $Z_1(t)$ is mixing.*

PROOF: Since $C_l(0, t) \rightarrow 0$ as $t \rightarrow \infty$, from Theorem 2 we get that the process must be mixing. ■

5.2 Type II fractional α -stable Ornstein-Uhlenbeck process

In this section we examine the asymptotic properties of the correlation cascade $C_l(0, t)$ for the process $Z_2(t)$ defined by (4.24). We verify the presence of long memory in this process and prove that $Z_2(t)$ is mixing.

Theorem 17. *Let $H > 0$ and $0 < \alpha < 2$. Then the correlation cascade of $Z_2(t)$ satisfies*

$$C_l(0, t) \sim \frac{e^{-t}}{(\Gamma(H - 1/\alpha + 1))^\alpha} \quad \text{as } t \rightarrow \infty. \quad (5.10)$$

PROOF: We have

$$C_l(0, t) = \frac{1}{(\Gamma(H - 1/\alpha + 1))^\alpha} \times \int_{-\infty}^{\infty} \min\{e^{-tH}(e^t - s)^{H-1/\alpha} \mathbf{1}_{(0, e^t)}(s) ; (1 - s)^{H-1/\alpha} \mathbf{1}_{(0, 1)}(s)\}^\alpha ds.$$

Consider first the case $H - 1/\alpha < 0$. Then, we obtain

$$C_l(0, t) = \frac{e^{-tH\alpha}}{(\Gamma(H - 1/\alpha + 1))^\alpha} \int_0^1 (e^t - s)^{H\alpha-1} ds.$$

Since for fixed $s \in (0, 1)$ we have

$$\frac{(e^t - s)^{H\alpha-1}}{e^{t(H\alpha-1)}} \longrightarrow 1$$

as $t \rightarrow \infty$, from the dominated convergence theorem we obtain

$$\int_0^1 (e^t - s)^{H\alpha-1} ds \sim e^{t(H\alpha-1)}.$$

Consequently,

$$C_l(0, t) \sim \frac{e^{-t}}{(\Gamma(H - 1/\alpha + 1))^\alpha} \quad \text{as } t \rightarrow \infty.$$

We pass to the second case $H - 1/\alpha > 0$. For $s \in (0, 1)$ we get

$$e^{-tH}(e^t - s)^{H-1/\alpha} < (1 - s)^{H-1/\alpha} \iff s < \frac{e^{-\frac{t}{H\alpha-1}} - 1}{e^{-\frac{tH\alpha}{H\alpha-1}} - 1}.$$

Set

$$k(t) := \frac{e^{-\frac{t}{H\alpha-1}} - 1}{e^{-\frac{tH\alpha}{H\alpha-1}} - 1}.$$

Then, we obtain

$$\begin{aligned} C_l(0, t) &= \frac{1}{(\Gamma(H - 1/\alpha + 1))^\alpha} \left(\int_0^{k(t)} e^{-tH\alpha}(e^t - s)^{H\alpha-1} ds + \int_{k(t)}^1 (1 - s)^{H\alpha-1} ds \right) \\ &=: \frac{1}{(\Gamma(H - 1/\alpha + 1))^\alpha} (I_1(t) + I_2(t)). \end{aligned}$$

For the first term $I_1(t)$, after some standard calculations, we get

$$I_1(t) = e^{-tH\alpha} k(t) \int_0^1 (e^t - k(t)s)^{H\alpha-1} ds.$$

Since $k(t) \longrightarrow 1$ and for fixed $s \in (0, 1)$

$$\frac{(e^t - k(t)s)^{H\alpha-1}}{e^{t(H\alpha-1)}} \longrightarrow 1$$

as $t \rightarrow \infty$, the dominated convergence theorem yields

$$I_1(t) \sim e^{-t} \quad \text{as } t \rightarrow \infty.$$

For the second term $I_2(t)$, we obtain

$$I_2(t) = \frac{(1 - k(t))^{H\alpha}}{H\alpha} \sim \frac{e^{-\frac{tH\alpha}{H\alpha-1}}}{H\alpha}$$

as $t \rightarrow \infty$. Thus, $I_2(t)$ decays faster than $I_1(t)$ and its contribution is negligible. Finally, we get

$$C_l(0, t) \sim \frac{e^{-t}}{(\Gamma(H - 1/\alpha + 1))^\alpha} \quad \text{as } t \rightarrow \infty. \blacksquare$$

We get the following conclusions

Corollary 12. *The process $Z_2(t)$ does not have long memory in the sense of (3.17).*

PROOF: The correlation cascade $C_l(0, t)$ decays exponentially, thus the series (3.17) converges. \blacksquare

Corollary 13. *The process $Z_2(t)$ is mixing.*

PROOF: Since $C_l(0, t) \rightarrow 0$ as $t \rightarrow \infty$, from Theorem 2 we get that the process is mixing. \blacksquare

5.3 Type III fractional α -stable Ornstein-Uhlenbeck process

In this section we investigate the asymptotic behaviour of the correlation cascade $C_l(0, t)$ for the process $Z_3(t)$ defined by (4.38). We show that $Z_3(t)$ does not have long memory in the sense of (3.17) and show that it is mixing.

Theorem 18. *Let $\kappa > 1 - 1/\alpha$ and $0 < \alpha < 2$. Then the correlation cascade of $Z_3(t)$ satisfies*

$$C_l(0, t) \sim \frac{(\Gamma(\kappa))^{-\alpha}}{\lambda\alpha} t^{\alpha(\kappa-1)} e^{-\lambda\alpha t} \quad \text{as } t \rightarrow \infty. \quad (5.11)$$

PROOF: Consider first the case $\kappa < 1$. We have

$$\begin{aligned} C_l(0, t) &= \Gamma(\kappa)^{-\alpha} \int_{-\infty}^{\infty} \min\{e^{-\lambda(t-s)}(t-s)^{\kappa-1} \mathbf{1}_{\{s < t\}} ; e^{\lambda s}(-s)^{\kappa-1} \mathbf{1}_{\{s < 0\}}\}^\alpha ds \\ &= \Gamma(\kappa)^{-\alpha} \int_{-\infty}^0 e^{-\lambda\alpha(t-s)}(t-s)^{\alpha(\kappa-1)} ds = \Gamma(\kappa)^{-\alpha} e^{-\lambda\alpha t} \int_0^{\infty} e^{-\lambda\alpha s}(t+s)^{\alpha(\kappa-1)} ds. \end{aligned}$$

For fixed $s \in (0, \infty)$ we have

$$\frac{e^{-\lambda\alpha s}(t+s)^{\alpha(\kappa-1)}}{t^{\alpha(\kappa-1)}} \longrightarrow e^{-\lambda\alpha s} \quad \text{as } t \rightarrow \infty.$$

Additionally,

$$\frac{e^{-\lambda\alpha s}(t+s)^{\alpha(\kappa-1)}}{t^{\alpha(\kappa-1)}} \leq c_1 e^{-\lambda\alpha s},$$

which is integrable on $(0, \infty)$. Here c_1 is the appropriate positive constant. Thus, from the dominated convergence theorem we get

$$C_l(0, t) \sim \Gamma(\kappa)^{-\alpha} t^{\alpha(\kappa-1)} e^{-\lambda\alpha t} \int_0^\infty e^{-\lambda\alpha s} ds = \frac{\Gamma(\kappa)^{-\alpha}}{\lambda\alpha} t^{\alpha(\kappa-1)} e^{-\lambda\alpha t}$$

as $t \rightarrow \infty$.

We pass to the case $\kappa > 1$. For $s < 0$ we have

$$e^{-\lambda t}(t-s)^{\kappa-1} < (-s)^{\kappa-1} \iff s < -\frac{t}{e^{\lambda t/(\kappa-1)} - 1}.$$

Set

$$k(t) := -\frac{t}{e^{\lambda t/(\kappa-1)} - 1}.$$

Then, we have

$$\begin{aligned} C_l(0, t) &= \Gamma(\kappa)^{-\alpha} \int_{-\infty}^\infty \min\{e^{-\lambda(t-s)}(t-s)^{\kappa-1} \mathbf{1}_{\{s < t\}}; e^{\lambda s}(-s)^{\kappa-1} \mathbf{1}_{\{s < 0\}}\}^\alpha ds \\ &= \Gamma(\kappa)^{-\alpha} \left(\int_{-\infty}^{k(t)} e^{-\lambda\alpha(t-s)}(t-s)^{\alpha(\kappa-1)} ds + \int_{k(t)}^0 e^{\lambda\alpha s}(-s)^{\alpha(\kappa-1)} ds \right) \\ &=: \Gamma(\kappa)^{-\alpha} (I_1(t) + I_2(t)). \end{aligned}$$

For the first term, after some standard calculations, we get

$$I_1(t) = e^{-\lambda\alpha t} \int_0^\infty h(s, t) ds,$$

where

$$h(s, t) = e^{-\lambda\alpha(k(t)+s)}(t+k(t)+s)^{\alpha(\kappa-1)}.$$

Additionally, for fixed $s \in (0, \infty)$, we obtain

$$h(s, t) \sim t^{\alpha(\kappa-1)} e^{-\lambda\alpha s} \quad \text{as } t \rightarrow \infty.$$

Consequently, from the dominated convergence theorem, we obtain

$$I_1(t) \sim \frac{1}{\lambda\alpha} t^{\alpha(\kappa-1)} e^{-\lambda\alpha t}$$

as $t \rightarrow \infty$.

For the second term we get

$$I_2(t) \leq -k(t)(-k(t))^{\alpha(\kappa-1)} = (-k(t))^{\alpha(\kappa-1)+1}.$$

Thus, $I_2(t)$ decays faster than $I_1(t)$ and its contribution is negligible. Finally, we obtain

$$C_l(0, t) \sim \Gamma(\kappa)^{-\alpha} I_2(t) \sim \frac{\Gamma(\kappa)^{-\alpha}}{\lambda \alpha} t^{\alpha(\kappa-1)} e^{-\lambda \alpha t}$$

as $t \rightarrow \infty$. ■

We get the following conclusions

Corollary 14. *The process $Z_3(t)$ does not have long memory in the sense of (3.17).*

PROOF: Since the correlation cascade $C_l(0, t)$ decays exponentially, the series (3.17) is convergent. ■

Corollary 15. *The process $Z_3(t)$ is mixing.*

PROOF: Since $C_l(0, t) \rightarrow 0$ as $t \rightarrow \infty$, from Theorem 2 we obtain that the process must be mixing. ■

5.4 Continuous-time FARIMA process

Since the integral representation of the introduced in Section 4.4 process $Z_4(t)$ is not known, we are unable to verify the behaviour of the corresponding correlation cascade. For this reason we pass to the continuous-time FARIMA process $Z_5(t)$ defined by (4.53). In what follows, we show that the process has long memory in the sense of (3.17) and prove that $Z_5(t)$ is mixing.

Theorem 19. *Let $\kappa < 1 - 1/\alpha$ and $0 < \alpha < 2$. Then the correlation cascade of $Z_5(t)$ satisfies*

$$C_l(0, t) \sim K \cdot t^{\alpha(\kappa-1)+1} \quad \text{as } t \rightarrow \infty. \quad (5.12)$$

Here K is the appropriate positive constant dependant only on the parameters α and κ .

PROOF: Since the function $f(s) = (s+\epsilon)^{\kappa-1} \mathbf{1}_{\{s>0\}}$ is non-increasing and positive, we get from formula (3.24)

$$C_l(0, t) = c \int_t^\infty (s+\epsilon)^{\alpha(\kappa-1)} ds,$$

where c is the appropriate positive constant. Therefore, we immediately obtain

$$C_l(0, t) \sim K \cdot t^{\alpha(\kappa-1)+1} \quad \text{as } t \rightarrow \infty$$

for appropriate positive constant K . ■

We get the following two corollaries

Corollary 16. *For $1 - \frac{2}{\alpha} \leq \kappa < 1 - \frac{1}{\alpha}$ the process $Z_5(t)$ has long memory in the sense of (3.17).*

PROOF: Since the correlation cascade of $Z_5(t)$ satisfies $C_l(0, t) \sim K \cdot t^{\alpha(\kappa-1)+1}$ as $t \rightarrow \infty$, the series (3.17) is divergent for $1 - \frac{2}{\alpha} \leq \kappa < 1 - \frac{1}{\alpha}$. ■

Corollary 17. *The process $Z_5(t)$ is mixing.*

PROOF: Since $C_l(0, t) \rightarrow 0$ as $t \rightarrow \infty$, from Theorem 2 we obtain that $Z_5(t)$ must be mixing. ■

Remark. Comparing the results for the asymptotic dependence structure of the processes $Z_i(t)$, $i = 1, 2, 3, 5$, obtained in Chapters 4 and 5, we see that the two alternative definitions of long memory given in (3.3) and (3.17) are equivalent for the discussed stationary processes. Moreover, the presence of long-range dependence in terms of the correlations in the Gaussian case transfers to the more general α -stable case.

Chapter 6

Conclusions

In the thesis we have obtained the following key results:

We have defined three new fractional generalizations of the standard α -stable O-U process and investigated their asymptotic dependence structure in terms of the generalized codifference. We have verified the presence of long-range dependence in these models.

We have proposed the Langevine-type fractional differential equation with α -stable noise, which is a continuous-time analogue of the FARIMA difference equation. We have shown that the process obtained as the solution of this differential equation has long memory. Moreover, it has exactly the same asymptotic dependence structure as FARIMA time series. Therefore, it can be regarded a continuous-time counterpart of FARIMA process. As a consequence, we have obtained the important relationship between the self-similarity index H , the index of stability α and the fractional parameter κ .

We have significantly simplified the classical Maruyama's mixing theorem for i.d. processes by reducing the number of necessary and sufficient conditions to only two. We have taken advantage of this result and proposed the description of ergodicity, weak mixing and mixing for i.d. processes in terms of the measure of dependence called correlation cascade. Next, using the techniques from ergodic theory, we have derived the relationship between two alternative measures of dependence for i.d. processes – correlation cascade and codifference.

We have used the correlation cascade to investigate the dependence structure and the ergodic properties of the previously introduced fractional processes. We have obtained the precise formulas for the asymptotic behaviour of the correlation cascade and verified the presence of long memory in the considered models. We have proved that the models are mixing. We have shown that both introduced definitions of long memory for α -stable processes, are equivalent for the discussed stationary fractional processes.

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