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Applications of Mathematics and Statistics in Economics

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SEARCH FOR CHAOS IN FINANCIAL TIME SERIES

1. Deterministic chaos in nonlinear systems

Even some simple **non-linear deterministic systems** can under certain conditions pass to chaotic states [1]. Chaotic behavior is bounded, not periodic and similar to random one. It is highly sensitive to small change of initial conditions and cannot be predicted for long time. It is called **deterministic chaos**.
As an simple example, let us consider discrete system described by **logistic difference equation**

$$x_{n+1} = Ax_n(1-x_n) = f(x_n),\ 0 < A \leqslant 4, \tag{1}$$

where A is control parameter. For $0 < A \leqslant 4$, values from the interval <0,1> will be mapped also into this interval. The function $f(x_n)$ is so called iterative function. The position of fixed points can be determined from the relation

$$\bar{x} = A\bar{x}(1-\bar{x}) \Rightarrow \bar{x}_1 = 0,\ \bar{x}_2 = 1 - \frac{1}{A}, \tag{2}$$

and their stability from the behavior of first derivatives

$$\frac{df}{dx}\Big|_{x=0} = A,\ \frac{df}{dx}\Big|_{x=1-1/A} = 2 - A. \tag{3}$$

For $A \leqslant 1$, there is only one fixed point $\bar{x}_1 = 0$, which is stable (attractor). A sequence of values $x_0, x_1, x_2, \ldots$ tends to converge to zero.

In the range $1 < A < 3$, two fixed points exist. Now, the point $\bar{x}_1 = 0$ is unstable (repellor) and $\bar{x}_2 = 1-(1/A)$ is stable (attractor). Trajectories from arbitrary initial condition converge to one-point attractor.

For $A=3$, the first bifurcation occurs and the second fixed point will be unstable as well. In the case of further increase of control parameter A, the function f^2 will have 4 fixed points. 2 of them correspond to unstable points of function f, another 2 ones are stable and correspond to periodic attractor of function f with period 2. For $A=3.45$, the trajectory with period 2 becomes unstable and stable trajectory with period 4 arises. This doubling of the period is repeated always at the growth of A and stable trajectories with periods 8, 16, 32, ... arise.

Let A_1 be the value of control parameter, which gives rise to period 2, A_2 is the value at which period 4 arises and A_n is the value at which period 2^n arises. Let us denote

$$\delta_n = \frac{A_n - A_{n-1}}{A_{n+1} - A_n}. \tag{4}$$

Feigenbaum has shown that this ratio is roughly the same for all n and it approaches to the limit

$$\delta = \lim_{n\to\infty} \delta_n = 4.66920161... \tag{5}$$

Thus, the intervals of the parameter A, at which bifurcation occurs, are still contracted in this ratio. The sequence of bifurcation values A_n, at which stable trajectories with period 2^n occur, is convergent and tends to the limit

$$A_\infty = \lim_{n\to\infty} A_n = 3.569946... \tag{6}$$

Within interval $A_\infty < A \leqslant 4$, the system behaviour becomes very complex. There are infinitely many intervals of the parameter A (periodic windows) with stable periodic trajectories. On the other hand, there are certain parameter values leading to chaotic behaviour. In the limit case $A=4$, also analytical solution exists in the form

$$x_n = \sin^2\left(2^n \arcsin\sqrt{x_0}\right). \tag{7}$$

Clearly, from this solution, obvious extreme sensitivity to very small changes of initial value x_0 is seen. Thus, the solution of logistic difference equation (1) leads in the case of increase of control parameter A from periodic solution with period 2 through bifurcation cascade of period doubling to chaotic behaviour.

Now we mention briefly another way of the emergence of a chaotic state. It is the case of the class of by part linear mapping. For example, to this class belongs symmetric roof mapping

$$\begin{aligned} x_{n+1} &= 2Ax_n \quad \text{for } 0 \leqslant x_n \leqslant 0.5, \\ x_{n+1} &= 2A(1-x_n) \quad \text{for } 0.5 \leqslant x_n \leqslant 1, \end{aligned} \tag{8}$$

with control parameter $0 < A \leqslant 1$. This function is continuous, but it has not the derivative at the point x=0.5. In the case $A < 0.5$, only one fixed point exists and namely $\bar{x} = 0$; this point is stable, because $2A < 1$. For $A > 0.5$, there are 2 fixed points

$$\bar{x}_1 = 0, \bar{x}_2 = \frac{2A}{2A+1} \tag{9}$$

and it holds

$$\left|\frac{df}{dx}(x = \bar{x}_1)\right| = \left|\frac{df}{dx}(x = \bar{x}_2)\right| = 2A > 1. \tag{10}$$

Both fixed points are unstable and trajectories are chaotic. In this case, chaotic behaviour arises suddenly for $A > 0.5$ and no bifurcations occur.

2. Quantification of chaotic behaviour

The reasons for the construction of quantitative characteristics of chaotic behaviour are the following:

- quantificators can help to distinguish deterministic chaos from “noisy” behaviour, produced by the action of external random influences,
- quantificators can help to determine minimum number of variables needed for the construction of a dynamical model of the system,
- quantificators can help to classify systems according to universally valid regularities,
- changes of quantificators may signalise changes in qualitative behaviour of a system.

In principle, we can use 2 kinds of description. The first type stresses the dynamics of chaotic behaviour and corresponding quantificators describe the system evolution and mutual position of neighbouring trajectories. The second type is based on the geometry of trajectories in state space. Here we shall confine ourselves to the first kind of the description.

One of characteristic features of chaotic behaviour is the divergence of neighbouring trajectories. Let us consider 2 close points $x_0, x_0 + \varepsilon$. Applying certain iterative function n-times, the distance between these points will be

$$d_n = \left| f^{(n)}(x_0 + \varepsilon) - f^{(n)}(x_0) \right|. \tag{11}$$

In the case of chaotic behaviour, we expect exponential growth of this distance with the number of iterations

$$\frac{d_n}{\varepsilon} = \frac{\left| f^{(n)}(x_0 + \varepsilon) - f^{(n)}(x_0) \right|}{\varepsilon} = \exp(\lambda n), \tag{12}$$

and thus

$$\lambda = \frac{1}{n} \ln \left(\frac{\left| f^{(n)}(x_0 + \varepsilon) - f^{(n)}(x_0) \right|}{\varepsilon} \right). \tag{13}$$

In limit approaching $\varepsilon \to 0$ we obtain after small algebra

$$\lambda = \frac{1}{n} \ln \left(\left| f'(x_0) \right| \left| f'(x_1) \right| \ldots \left| f'(x_n) \right| \right) \tag{14}$$

where the dash denotes the derivative with respect to x. In alternative expression

$$\lambda = \frac{1}{n} \left(\ln \left| f'(x_0) \right| + \ln \left| f'(x_1) \right| + \ldots + \ln \left| f'(x_n) \right| \right). \tag{15}$$

This Lyapunov exponent characterizes the speed of the divergence of neighbouring trajectories. It is given as averaged natural logarithm of absolute value of the derivatives of iterative function at individual points of a trajectory. One-dimensional iterative function has chaotic trajectories, if average Lyapunov exponent is positive (the condition of divergence).

Having a time series of equidistant values $x_0, x_1, x_2, \ldots, x_n$, we can determine Lyapunov exponent using the following approach. Let us take two trajectories with starting points x_i, x_j and let us create the sequence of differences

$$d_0 = \left| x_j - x_i \right| \quad d_1 = \left| x_{j+1} - x_{i+1} \right| \qquad d_n = \left| x_{j+n} - x_{i+n} \right|. \tag{16}$$

We shall assume time evolution in the form

$$d_n = d_0 \exp(\lambda n) \Rightarrow \lambda = \frac{1}{n} \ln \frac{d_n}{d_0}. \tag{17}$$

For $\lambda > 0$, the behaviour of trajectories is chaotic. However, generally, λ depends on the choice of x_i. Therefore, the more reliable is to compute average Lyapunov coefficient from large number N initial values regularly distributed over whole attractor. Then we get

$$\bar{\lambda} = \frac{1}{N}\sum_{i=1}^{N}\lambda(x_i). \tag{18}$$

In the case of logistic function with control parameter A=4 is average Lyapunov exponent given by analytical expression

$$\bar{\lambda} = \int_0^1 \frac{\ln|4(1-2x)|}{\sqrt{x(1-x)}}dx = \ln 2. \tag{19}$$

Grassberger and Procaccia introduced the characteristic called correlation dimension, based on the behaviour of so called correlation sum [3]. For the computation of correlation dimension, we need the data about the evolution of a trajectory (in sum n values). For each i-th point of the trajectory, we seek relative frequency $p_i(r)$ of trajectory points at distance less than r from the point i (except i-th point)

$$p_i(r) = \frac{n_i}{n-1}. \tag{20}$$

Correlation sum is then computed as average relative frequency

$$C_1(r) = \frac{1}{n}\sum_{i=1}^{n}p_i(r) = \frac{1}{n(n-1)}\sum_{i=1}^{n}\sum_{j=1,j\neq i}^{n}H\left(r-|x_i-x_j|\right), \tag{21}$$

where H denotes Heaviside step function. Obviously $C_1(r)$=0, if r is less than minimal distance among the points of a trajectory. On contrary $C_1(r)$=1 means the distances among individual points do not exceed r. Minimal possible non-zero value $C_1(r)=2/(n(n-1))$ occurs in the case, only one distance is less than r.

In limit case $n \to \infty$, correlation sum is melted into correlation integral. Correlation dimension is then given by formula

$$D_1 = \lim_{r\to 0}\frac{\log C_1(r)}{\log r}. \tag{22}$$

A time series of single variable can be often sufficient for the determination of important characteristics of a multidimensional dynamical system. The groups of values

$$x_{t+1}, x_{t+2}, \ldots, x_{t+d} \quad t = 0, 1, 2, \ldots, (n-d) \tag{23}$$

give the coordinates of a point in d-dimensional space. Then the sequence of these groups describes the time evolution of a system in d-dimensional embedding space. In this case, correlation sum can be written as

$$C_d(r) = \frac{1}{n(n-1)} \sum_{i=1}^{n} \sum_{j=1, j \neq i}^{n} H\left(r - \left|\mathbf{x_i} - \mathbf{x_j}\right|\right), \tag{24}$$

because it depends on embedded dimension d. Vector $\mathbf{x_i}$ of dimension d has components

$$\mathbf{x_i} = \left(x_i, x_{i+L}, x_{i+2L}, \ldots, x_{i+(d-1)L}\right), \tag{25}$$

where L is time lag between neighbouring values. The length of the difference of two vectors is mostly calculated as Euclid distance

$$\left|\mathbf{x_i} - \mathbf{x_j}\right| = \sqrt{\sum_{k=0}^{d-1} \left(x_{i+kL} - x_{j+kL}\right)^2}. \tag{26}$$

Then it holds for correlation dimension

$$D_d = \lim_{r \to 0} \frac{\log C_d(r)}{\log r}. \tag{27}$$

In the case of i.i.d. (independent identically distributed) process with regular distribution is

$$D_1 = \lim_{r \to 0} \frac{\log C_1(r)}{\log r} = \lim_{r \to 0} \frac{\log 2 + \log r}{\log r} = 1, \tag{28}$$

$$D_2 = \lim_{r \to 0} \frac{\log C_2(r)}{\log r} = \lim_{r \to 0} \frac{\log 4 + 2\log r}{\log r} = 2 \tag{29}$$

and generally $D_d = d$. On the contrary, in the case of non-linear deterministic process is the behaviour of correlation sum quite different. For example, in the case of roof mapping is $D_1 = D_2 = \ldots = D_d = 1$.

Brock, Dechert and Scheinkman showed, for a finite r and i.i.d. process, the following relation is valid [2]

$$C_d(r) = [C_1(r)]^d \tag{30}$$

and they suggested test statistic

$$T_d(r,n) = \frac{C_d(r,n) - \left[C_1(r,n)\right]^d}{s_d(r,n)}, \tag{31}$$

where $C_d(r,n), C_1(r,n)$ are sample correlation sums and $s_d(r,n)$ is the estimate of the standard deviation of the difference in expression (31). This statistic has asymptotically standard normal distribution $N(0,1)$ providing the validity of null hypothesis (i.i.d. process).

3. Application

Let us apply now above mentioned techniques both to model and financial data. First, consider logistic difference equation (1) with control parameter $A = 4$ and simulate a time series with length size $n = 100$. Clearly, this time series exhibits chaotic behaviour in the form of irregular oscillations. Nevertheless, the corresponding attractor has parabolical shape and correlation dimension equal to 1. The results of calculation are given in Table 1.

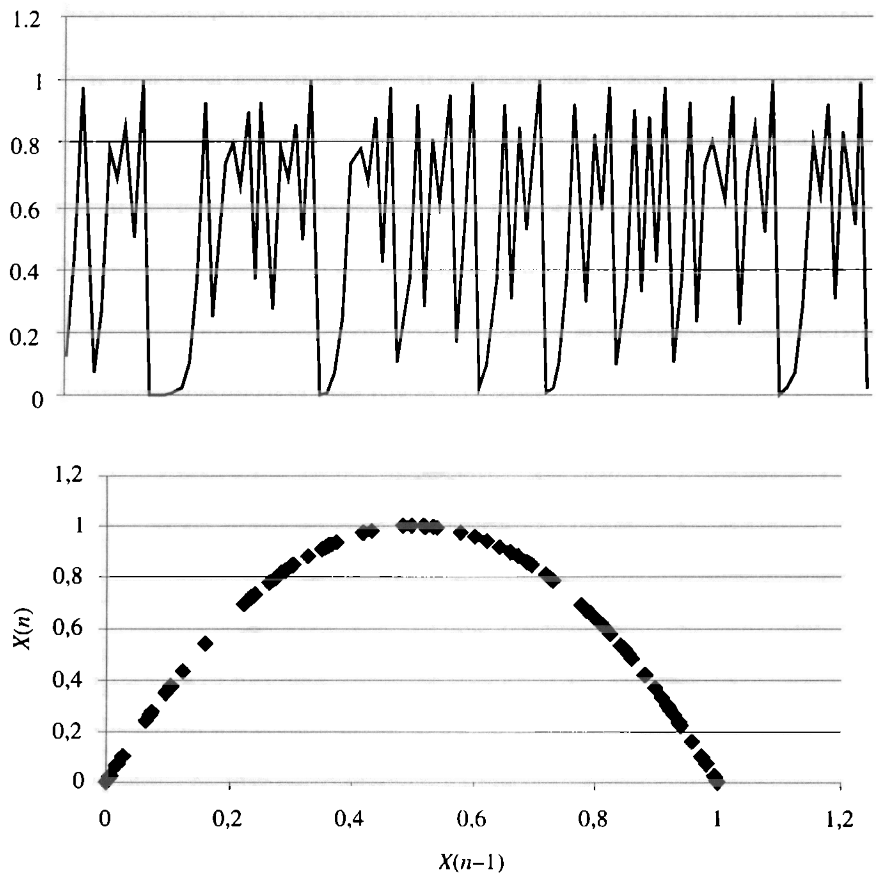

Fig. 1. Logistic difference equation: time series (above), X_n versus X_{n-1} plot (below), $n = 100$

Table 1. Logistic difference equation: correlation dimension computed for $d = 1, 2, \ldots, 6$, $n = 2000$

d	1	2	3	4	5	6
$C(d)$	0.940	0.960	1.001	1.008	1.037	1.059

Clearly, correlation dimension $C(d)$ is independent on embedded dimension d; moderate fluctuations around correct value $C(d) = 1$ are due to finite sample size.

As the second example, let us consider symmetric roof mapping in the form

$$X_{n+1} = 1 - 2|X_n| \tag{32}$$

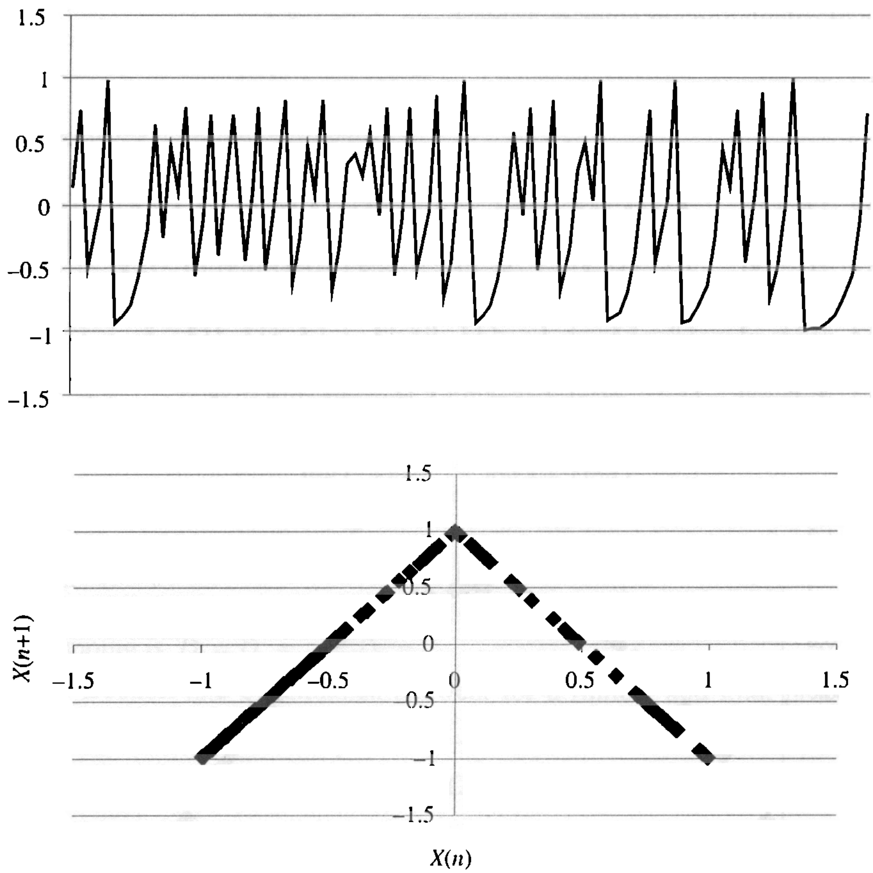

Fig. 2. Symmetric roof mapping: time series (above), X_n versus X_{n-1} plot (below), $n = 100$

Again, the correlation dimension should be $C(d) = 1$; the computed values are in Table 2.

Table 2. Symmetric roof mapping: correlation dimension computed for $d = 1, 2, \ldots, 6$, $n = 2000$

d	1	2	3	4	5	6
$C(d)$	1.018	1.021	1.030	1.050	1.082	1.175

Third, as the opposite case, consider normal white noise $N(0,1)$. The pattern of correlation dimension is as in Table 3 and there is tendency to satisfy $C(d) = d$.

Table 3. Normal white noise $N(0,1)$: correlation dimension computed for $d = 1, 2, \ldots, 6$, $n = 2000$

d	1	2	3	4	5	6
$C(d)$	1.017	2.081	3.069	3.855	4.555	5.140

Further, we have investigated selected data from Czech capital market. Input data are given as stock price time series x_t during 1999-2003, i.e. 1254 daily values for each series. The following stocks belonging to "blue chips" in Prague Stock Exchange were selected and analysed: *ČEZ* (CEZ), *Komerční banka* (KB), *České radiokomunikace* (RAD), *Tabak* (TAB), *Český Telecom* (TEL) and *Unipetrol* (UNIP). The subject of our analysis were logarithmic returns (further only returns) expressed as percentage and computed as

$$y_t = 100\left(\ln x_t - \ln x_{t-1}\right). \tag{33}$$

First, the correlation dimensions were computed and results compiled in Table 4.

Table 4. Correlation dimensions of stock returns time series

d	1	2	3	4	5	6
CEZ	0.658	1.850	2.959	3.673	4.601	4.854
KB	0.765	1.951	3.024	3.784	4.484	4.930
RAD	0.565	1.825	2.919	3.652	4.217	4.758
TEL	0.920	1.992	2.946	3.806	4.282	4.543
UNIP	0.582	1.883	2.999	3.693	4.500	5.183

One can see only little difference in comparison with white noise. Clearly, the corresponding embedding dimension should be rather high. Further, BDS test (31) was performed with the results given in Table 5.

Table 5. Test statistics for BDS test. Critical value = 1.96

d	2	3	4	5	6
CEZ	8.30	9.76	10.50	11.09	11.44
KB	9.25	9.54	10.53	11.36	12.17
RAD	8.88	10.48	12.13	12.83	13.59
TAB	6.83	8.96	9.39	9.78	10.25
TEL	6.49	7.94	9.10	9.90	10.80
UNIP	8.60	10.56	12.03	13.38	14.52

All test statistics markedly exceed critical value. Thus, despite the similarity of correlation dimensions, stock returns do not create an i.i.d. process.

Last, we have applied so called rescaled range analysis. The general procedure is the following:

- given a time series containing values $x_1, x_2, ..., x_N$ registered in time points $t = 1, 2, ..., N$; thus, N time intervals with unit length arises;
- subdivide the whole time series into m neighbouring not overlapping intervals with length n, so that $N = mn$;
- compute average value for each interval

$$\bar{x}_j = \frac{1}{n}\sum_{i=1}^{n} x_{ij} \quad j = 1,2,...,m; \tag{34}$$

- create the time series of cumulative deviations from average value for each interval

$$z_{kj} = \sum_{i=1}^{k}\left(x_{ij} - \bar{x}_j\right) \quad k = 1,2,...,n; \tag{35}$$

- compute the range of cumulated deviations for each interval

$$R_j = \max\left(z_{kj}\right) - \min\left(z_{kj}\right) \geqslant 0; \tag{36}$$

- compute standard deviation for each interval

$$s_j = \sqrt{\frac{1}{n-1}\sum_{i=1}^{n}\left(x_{ij} - \bar{x}_j\right)^2}; \tag{37}$$

- compute rescaled range R_j / s_j for each interval;
- average value R/S for an interval with length n is then

$$\left(R/S\right)_n = \frac{1}{m}\sum_{j=1}^{m}\left(R_j / s_j\right). \tag{38}$$

Hurst assumed general type of the dependence of R/S on time in the form

$$\left(R/S\right)_n = Cn^H, \tag{39}$$

where C is a constant and H **Hurst exponent**. In practice, H is estimated using linear regression

$$\log\left[\left(R/S\right)_n\right] = \log C + H \log n \tag{40}$$

by virtue of the values $(R/S)_n$ computed for different n. Interpretation of Hurst exponent is the following.

- If H=0.50, then is time series generated by i.i.d. process. *R/S* analysis is a non-parametric procedure and it does not demand any assumptions about process parameters.
- In the range $0.50 < H < 1.00$ are corresponding processes called persistent and they are characterised by long memory. There is no characteristic time scale, which is typical for fractal time series. They are suitable for the modelling of time series of stock returns.
- In the range $0 < H < 0.50$ so called antipersistent processes are underlying, which alter signs more frequently than white noise. They are suitable for the modelling of volatility in financial time series.

Hurst exponents computed are compilled in the following Table 6.

Table 6. Hurst exponents

CEZ	KB	RAD	TAB	TEL	UNIP
0.516	0.535	0.558	0.452	0.518	0.527

Thus, in 5 cases are Hurst exponents slightly higher than 0.5 and there is weak tendency to persistent behaviour, i.e. cycles creation. The only exception represents TAB stock with Hurst exponent less than 0.5.

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POSZUKIWANIE CHAOSU W FINANSOWYCH SZEREGACH CZASOWYCH

Streszczenie

Zachowanie chaotyczne może być generowane w nieliniowych systemach dynamicznych pod pewnymi warunkami. Jest regułą, że nie jesteśmy zdolni rozwiązać odpowiednich nieliniowych równań różniczkowych i jesteśmy nieświadomi co do warunków początkowych. Jednocześnie dla danego szeregu czasowego jesteśmy w stanie obliczyć pewne ważne charakterystyki, takie jak wykładnik Lapunowa, wykładnik Hursta i wymiar korelacji. W celu rozróżnienia między chaosem deterministycznym a procesem niezależnym o identycznym rozkładzie można wykorzystać test BDS. Artykuł jest próbą analizy szeregów czasowych z czeskiego rynku kapitałowego. Stosując wspomniane metody, odkryto pewien rodzaj trwałego zachowania, tj. słabą tendencję do tworzenia cykli.

Słowa kluczowe: nieliniowe systemy dynamiczne, chaos deterministyczny, finansowe szeregi czasowe.